CANADIAN EDUCATIONAL SERIES

TODHUNTER'S

EUCLID

FOR SCHOOLS & COLLEGES

BOOKS I., II., III.

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Canadian Series of School Books.

THE ELEMENTS OF

EUCLID

FOR THE USE OF SCHOOLS AND COLLEGES.

BOOKS I., II., III.

BY

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EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. A POINT is that which has no parts, or which has no magnitude.

2. A line is length without breadth.

3. The extremities of a line are points.

4. A straight line is that which lies evenly between its extreme points.

5. A superficies is that which has only length and breadth.

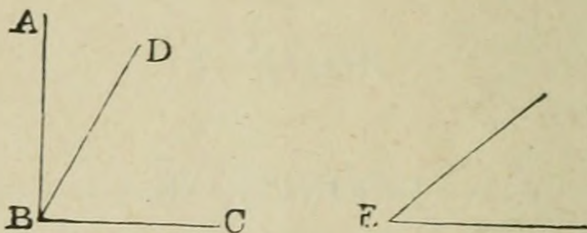
6. The extremities of a superficies are lines.

7. A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

8. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

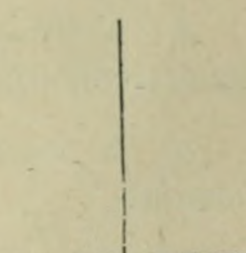
9. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

Note. When several angles are at one point B, any one of them is expressed by three letters, of which the letter which is at the vertex of the angle, that is, at the point at which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two letters is somewhere on one of those straight lines, and the other letter on the other straight line. Thus the angle which is contained by the

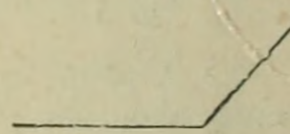


straight lines AB, CB is named the angle ABC, or CBA; the angle which is contained by the straight lines AB, DB is named the angle ABD, or DBA; and the angle which is contained by the straight lines DB, CB is named the angle DBC, or CBD; but if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at E.

10. When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



11. An obtuse angle is that which is greater than a right angle.

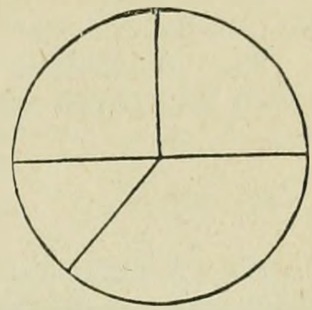


12. An acute angle is that which is less than a right angle.

13. A term or boundary is the extremity of any thing.

14. A figure is that which is enclosed by one or more boundaries.

15. A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another:



16. And this point is called the centre of the circle.

17. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

[A radius of a circle is a straight line drawn from the centre to the circumference.]

18. A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. A segment of a circle is the figure contained by a straight line and the circumference which it cuts off.

20. Rectilineal figures are those which are contained by straight lines:

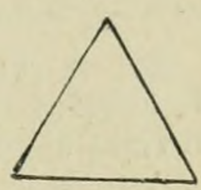
21. Trilateral figures, or triangles, by three straight lines:

22. Quadrilateral figures by four straight lines:

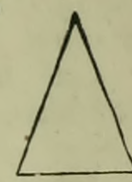
23. Multilateral figures, or polygons, by more than four straight lines.

24. Of three-sided figures,

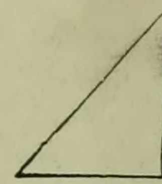
An equilateral triangle is that which has three equal sides:



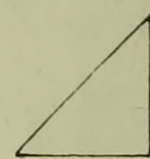
25. An isosceles triangle is that which has two sides equal:



26. A scalene triangle is that which has three unequal sides:

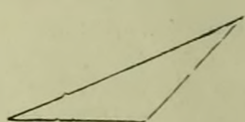


27. A right-angled triangle is that which has a right angle:

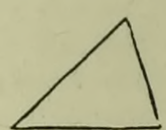


[The side opposite to the right angle in a right-angled triangle is frequently called the hypotenuse.]

28. An obtuse-angled triangle is that which has an obtuse angle:

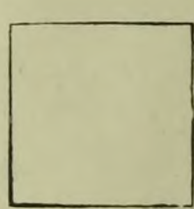


29. An acute-angled triangle is that which has three acute angles.

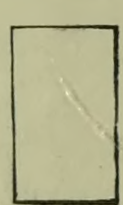


Of four-sided figures,

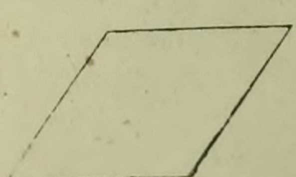
30. A square is that which has all its sides equal, and all its angles right angles:



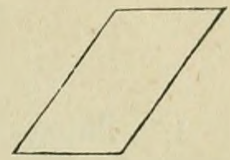
31. An oblong is that which has all its angles right angles, but not all its sides equal:



32. A rhombus is that which has all its sides equal, but its angles are not right angles:

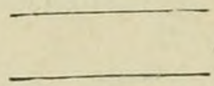


33. A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles :



34. All other four-sided figures besides these are called trapeziums.

35. Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.



[*Note.* The terms *oblong* and *rhomboid* are not often used. Practically the following definitions are used. Any four-sided figure is called a *quadrilateral.* A line joining two opposite angles of a quadrilateral is called a *diagonal.* A quadrilateral which has its opposite sides parallel is called a *parallelogram.* The words *square* and *rhombus* are used in the sense defined by Euclid ; and the word *rectangle* is used instead of the word *oblong.*

Some writers propose to restrict the word *trapezium* to a quadrilateral which has two of its sides parallel ; and it would certainly be convenient if this restriction were universally adopted.]

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point :

2. That a terminated straight line may be produced to any length in a straight line :

3. And that a circle may be described from any centre, at any distance from that centre.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

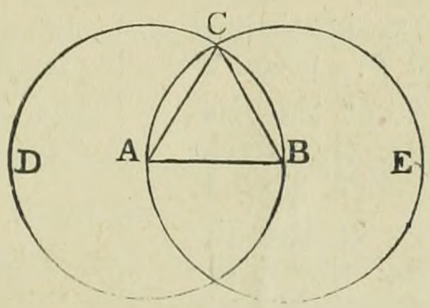
11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

PROPOSITION 1. *PROBLEM.*

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line: it is required to describe an equilateral triangle on AB.



From the centre A, at the distance AB, describe the circle BCD. [*Postulate* 3.

From the centre B, at the distance BA, describe the circle ACE. [*Postulate* 3.

From the point C, at which the circles cut one another, draw the straight lines CA and CB to the points A and B. [*Post.* 1.

ABC shall be an equilateral triangle.

Because the point A is the centre of the circle BCD, AC is equal to AB. [*Definition* 15.

And because the point B is the centre of the circle ACE, BC is equal to BA. [*Definition* 15.

But it has been shewn that CA is equal to AB;

therefore CA and CB are each of them equal to AB.

But things which are equal to the same thing are equal to one another. [*Axiom* 1.

Therefore CA is equal to CB.

Therefore CA, AB, BC are equal to one another.

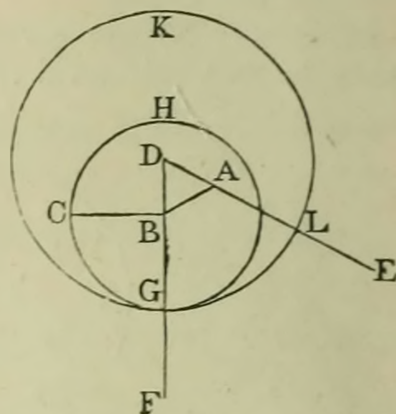
Wherefore *the triangle ABC is equilateral,* [*Def.* 24.

and it is described on the given straight line AB. Q.E.F.

PROPOSITION 2. *PROBLEM.*

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line: it is required to draw from the point A a straight line equal to BC.



From the point A to B draw the straight line AB; [*Post.* 1.

and on it describe the equilateral triangle DAB, [I. 1.

and produce the straight lines DA, DB to E and F. [*Post.* 2.

From the centre B, at the distance BC, describe the circle CGH, meeting DF at G. [*Post.* 3.

From the centre D, at the distance DG, describe the circle GKL, meeting DE at L. [*Post.* 3.

AL shall be equal to BC.

Because the point B is the centre of the circle CGH, BC is equal to BG. [*Definition* 15.

And because the point D is the centre of the circle GKL, DL is equal to DG; [*Definition* 15.

and DA, DB parts of them are equal; [*Definition* 24.

therefore the remainder AL is equal to the remainder BG. [*Axiom* 3.

But it has been shewn that BC is equal to BG;

therefore AL and BC are each of them equal to BG.

But things which are equal to the same thing are equal to one another. [*Axiom* 1.

Therefore AL is equal to BC.

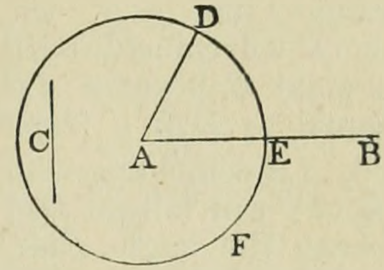
Wherefore *from the given point A a straight line AL has been drawn equal to the given straight line BC.* Q.E.F.

PROPOSITION 3. *PROBLEM.*

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which

AB is the greater: it is required to cut off from *AB*, the greater, a part equal to *C* the less.



From the point *A* draw the straight line *AD* equal to *C*; [I. 2.

and from the centre *A*, at the distance *AD*, describe the circle *DEF* meeting *AB* at *E*. [*Postulate* 3.

AE shall be equal to *C*.

Because the point *A* is the centre of the circle *DEF*, *AE* is equal to *AD*. [*Definition* 15.

But *C* is equal to *AD*. [*Construction.*

Therefore *AE* and *C* are each of them equal to *AD*.

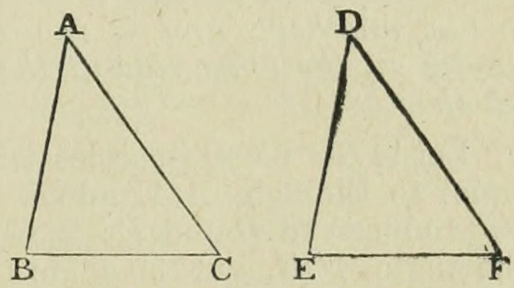
Therefore *AE* is equal to *C*. [*Axiom* 1.

Wherefore *from AB the greater of two given straight lines a part AE has been cut off equal to C the less.* Q.E.F.

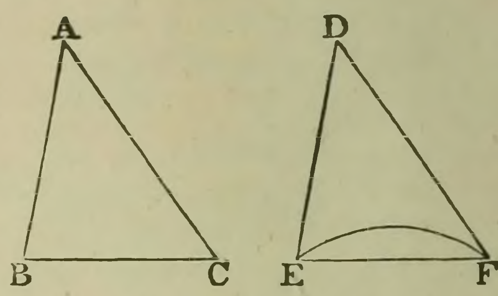
PROPOSITION 4. *THEOREM.*

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

Let *ABC*, *DEF* be two triangles which have the two sides *AB*, *AC* equal to the two sides *DE*, *DF*, each to each, namely, *AB* to *DE*, and *AC* to *DF*, and the angle *BAC* equal to the angle *EDF*: the base *BC* shall be equal to the base *EF*, and the triangle *ABC* to the triangle *DEF*, and the other angles shall be equal, each to each, to which the equal sides are opposite, namely, the angle *ABC* to the angle *DEF*, and the angle *ACB* to the angle *DFE*.



For if the triangle ABC be applied to the triangle DEF, so that the point A may be on the point D, and the straight line AB on the straight line DE, the point B will coincide with the point E, because AB is equal to DE. [*Hyp.*



And, AB coinciding with DE, AC will fall on DF, because the angle BAC is equal to the angle EDF. [*Hypothesis.*

Therefore also the point C will coincide with the point F, because AC is equal to DF. [*Hypothesis.*

But the point B was shewn to coincide with the point E, therefore the base BC will coincide with the base EF;

because, B coinciding with E and C with F, if the base BC does not coincide with the base EF, two straight lines will enclose a space; which is impossible. [*Axiom* 10.

Therefore the base BC coincides with the base EF, and is equal to it. [*Axiom* 8.

Therefore the whole triangle ABC coincides with the whole triangle DEF, and is equal to it. [*Axiom* 8.

And the other angles of the one coincide with the other angles of the other, and are equal to them, namely, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

Wherefore, *if two triangles* &c. Q.E.D.

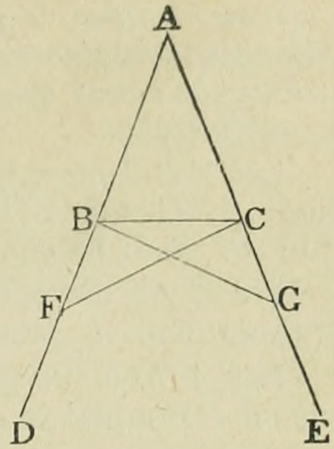
PROPOSITION 5. *THEOREM.*

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

Let ABC be an isosceles triangle, having the side AB equal to the side AC, and let the straight lines AB, AC be produced to D and E: the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F,

and from AE the greater cut off AG equal to AF the less, [I.3.



and join FC, GB.

Because AF is equal to AG, [*Constr.*
and AB to AC, [*Hypothesis.*
the two sides FA, AC are equal to the two sides GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB;
therefore the base FC is equal to the base GB, and the triangle AFC to the triangle AGB, and the remaining angles of the one to the remaining angles of the other, each to each, to which the equal sides are opposite, namely the angle ACF to the angle ABG, and the angle AFC to the angle AGB. [I. 4.

And because the whole AF is equal to the whole AG, of which the parts AB, AC are equal, [*Hypothesis.*
the remainder BF is equal to the remainder CG. [*Axiom* 3.
And FC was shewn to be equal to GB;
therefore the two sides BF, FC are equal to the two sides CG, GB, each to each;
and the angle BFC was shewn to be equal to the angle CGB;
therefore the triangles BFC, CGB are equal, and their other angles are equal, each to each, to which the equal sides are opposite, namely the angle FBC to the angle GCB, and the angle BCF to the angle CBG. [I. 4.

And since it has been shewn that the whole angle ABG is equal to the whole angle ACF,
and that the parts of these, the angles CBG, BCF are also equal;
therefore the remaining angle ABC is equal to the remaining angle ACB, which are the angles at the base of the triangle ABC. [*Axiom* 3.

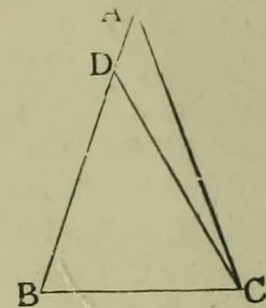
And it has also been shewn that the angle FBC is equal to the angle GCB, which are the angles on the other side of the base.

Wherefore, *the angles* &c. Q.E.D.

Corollary. Hence every equilateral triangle is also equiangular.

PROPOSITION 6. *THEOREM.*

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.



Let ABC be a triangle, having the angle ABC equal to the angle ACB: the side AC shall be equal to the side AB.

For if AC be not equal to AB, one of them must be greater than the other.

Let AB be the greater, and from it cut off DB equal to AC the less, [I. 3.
and join DC.

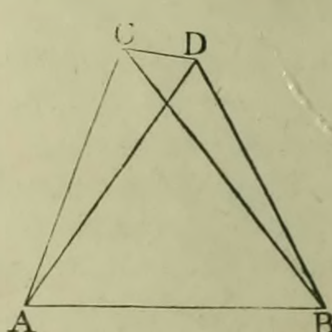
Then, because in the triangles DBC, ACB,
DB is equal to AC, [*Construction.*
and BC is common to both,
the two sides DB, BC are equal to the two sides AC, CB, each to each;
and the angle DBC is equal to the angle ACB; [*Hypothesis.*
therefore the base DC is equal to the base AB, and the triangle DBC is equal to the triangle ACB, [I. 4.
the less to the greater; which is absurd. [*Axiom* 9.
Therefore AB is not unequal to AC, that is, it is equal to it.

Wherefore, *if two angles* &c. Q.E.D.

Corollary. Hence every equiangular triangle is also equilateral.

PROPOSITION 7. *THEOREM.*

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.



If it be possible, on the same base AB, and on the same side of it, let there be two triangles ACB, ADB, having their sides CA, DA, which are terminated at the extremity A of the base, equal

to one another, and likewise their sides CB, DB, which are terminated at B equal to one another.

Join CD. In the case in which the vertex of each triangle is without the other triangle;

because AC is equal to AD, [*Hypothesis*.

the angle ACD is equal to the angle ADC. [I. 5.

But the angle ACD is greater than the angle BCD, [*Ax.* 9.

therefore the angle ADC is also greater than the angle BCD;

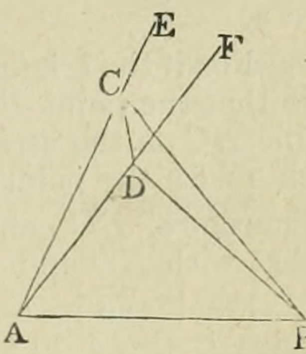
much more then is the angle BDC greater than the angle BCD.

Again, because BC is equal to BD, [*Hypothesis*.

the angle BDC is equal to the angle BCD. [I. 5.

But it has been shewn to be greater; which is impossible.

But if one of the vertices as D, be within the other triangle ACB, produce AC, AD to E, F.



Then because AC is equal to AD, in the triangle ACD, [*Hyp.*

the angles ECD, FDC, on the other side of the base CD, are equal to one another. [I. 5.

But the angle ECD is greater than the angle BCD, [*Axiom* 9.

therefore the angle FDC is also greater than the angle BCD;

much more then is the angle BDC greater than the angle BCD.

Again, because BC is equal to BD, [*Hypothesis*.

the angle BDC is equal to the angle BCD. [I. 5.

But it has been shewn to be greater; which is impossible.

The case in which the vertex of one triangle is on a side of the other needs no demonstration.

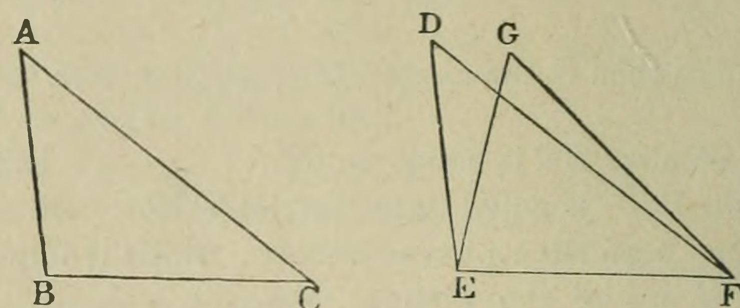
Wherefore, *on the same base* &c. Q.E.D.

PROPOSITION 8. *THEOREM.*

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their

bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC equal to the two sides DE, DF, each to each, namely AB to DE, and AC to DF, and also the base BC equal to the base EF: the angle BAC shall be equal to the angle EDF.



For if the triangle ABC be applied to the triangle DEF, so that the point B may be on the point E, and the straight line BC on the straight line EF, the point C will also coincide with the point F, because BC is equal to EF. [*Hyp.*
Therefore, BC coinciding with EF, BA and AC will coincide with ED and DF.

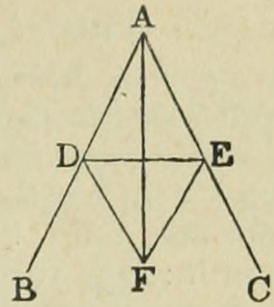
For if the base BC coincides with the base EF, but the sides BA, CA do not coincide with the sides ED, FD, but have a different situation as EG, FG; then on the same base and on the same side of it there will be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise their sides which are terminated at the other extremity.
But this is impossible. [I. 7.
Therefore since the base BC coincides with the base EF, the sides BA, AC must coincide with the sides ED, DF. Therefore also the angle BAC coincides with the angle EDF, and is equal to it. [*Axiom* 8.

Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 9. *PROBLEM.*

To bisect a given rectilineal angle, that is to divide it into two equal angles.

Let BAC be the given rectilineal angle: it is required to bisect it.



Take any point D in AB, and from AC cut off AE equal to AD; [I. 3.

join DE, and on DE, on the side remote from A, describe the equilateral triangle DEF. [I. 1.

Join AF. The straight line AF shall bisect the angle BAC.

Because AD is equal to AE, [*Construction.*
and AF is common to the two triangles DAF, EAF,
the two sides DA, AF are equal to the two sides EA, AF, each to each;

and the base DF is equal to the base EF; [*Definition* 24.

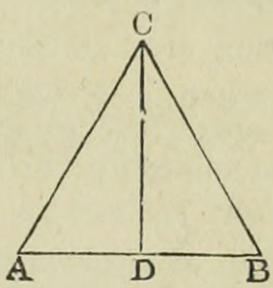
therefore the angle DAF is equal to the angle EAF. [I. 8.

Wherefore *the given rectilineal angle BAC is bisected by the straight line AF.* Q.E.F.

PROPOSITION 10. *PROBLEM.*

To bisect a given finite straight line, that is to divide it into two equal parts.

Let AB be the given straight line: it is required to divide it into two equal parts.



Describe on it an equilateral triangle ABC, [I. 1.

and bisect the angle ACB by the straight line CD, meeting AB at D. [I. 9.

AB shall be cut into two equal parts at the point D.

Because AC is equal to CB, [*Definition* 24.
and CD is common to the two triangles ACD, BCD,
the two sides AC, CD are equal to the two sides BC, CD, each to each;

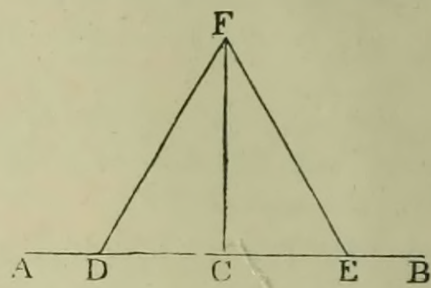
and the angle ACD is equal to the angle BCD; [*Constr.*

therefore the base AD is equal to the base DB. [I. 4.

Wherefore *the given straight line AB is divided into two equal parts at the point D.* Q.E.F.

PROPOSITION 11. *PROBLEM.*

To draw a straight line at right angles to a given straight line, from a given point in the same.



Let AB be the given straight line, and C the given point in it: it is required to draw from the point C a straight line at right angles to AB.

Take any point D in AC, and make CE equal to CD. [I. 3
On DE describe the equilateral triangle DFE, [I. 1.
and join CF.

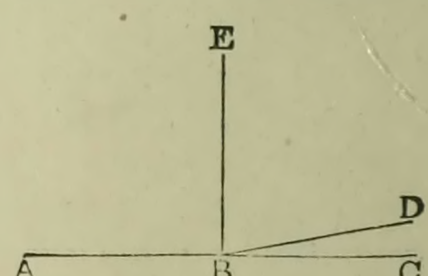
The straight line CF drawn from the given point C shall be at right angles to the given straight line AB.

Because DC is equal to CE, [*Construction.*
and CF is common to the two triangles DCF, ECF;
the two sides DC, CF are equal to the two sides EC, CF, each to each;
and the base DF is equal to the base EF; [*Definition* 24.
therefore the angle DCF is equal to the angle ECF; [I. 8.
and they are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [*Definition* 10.
therefore each of the angles DCF, ECF is a right angle.

Wherefore *from the given point C in the given straight line AB, CF has been drawn at right angles to AB.* Q.E.F.

Corollary. By the help of this problem it may be shewn that two straight lines cannot have a common segment.



If it be possible, let the two straight lines ABC, ABD have the segment AB common to both of them.

From the point B draw BE at right angles to AB.

Then, because ABC is a straight line, [*Hypothesis.*
the angle CBE is equal to the angle EBA [*Definition* 10.

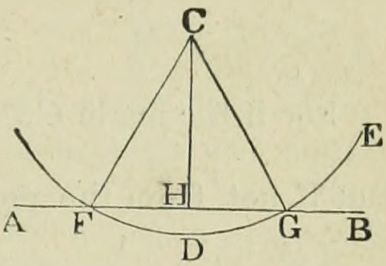
Also, because ABD is a straight line, [*Hypothesis.*
the angle DBE is equal to the angle EBA.
Therefore the angle DBE is equal to the angle CBE, [*Ax.* 1.
the less to the greater; which is impossible. [*Axiom* 9.

Wherefore *two straight lines cannot have a common segment.*

PROPOSITION 12. *PROBLEM.*

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be the given point without it: it is required to draw from the point C a straight line perpendicular to AB.



Take any point D on the other side of AB, and from the centre C, at the distance CD, describe the circle EGF, meeting AB at F and G. [*Postulate* 3.
Bisect FG at H, [I. 10.
and join CH.

The straight line CH drawn from the given point C shall be perpendicular to the given straight line AB.

Join CF, CG

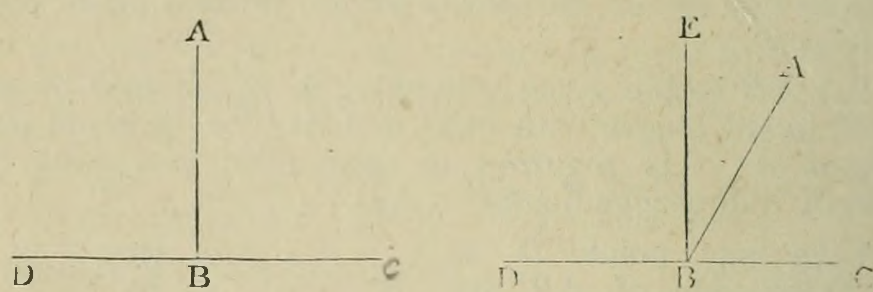
Because FH is equal to HG, [*Construction.*
and HC is common to the two triangles FHC, GHC;
the two sides FH, HC are equal to the two sides GH, HC, each to each;
and the base CF is equal to the base CG; [*Definition* 15.
therefore the angle CHF is equal to the angle CHG; [I. 8.
and they are adjacent angles
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it. [*Def.* 10.

Wherefore *a perpendicular CH has been drawn to the given straight line AB from the given point C without it.* Q.E.F.

PROPOSITION 13. *THEOREM.*

The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

Let the straight line AB make with the straight line CD, on one side of it, the angles CBA, ABD: these either are two right angles, or are together equal to two right angles.



For if the angle CBA is equal to the angle ABD, each of them is a right angle. [*Definition* 10.

But if not, from the point B draw BE at right angles to CD; [I. 11.

therefore the angles CBE, EBD are two right angles. [*Def.* 10.

Now the angle CBE is equal to the two angles CBA, ABE; to each of these equals add the angle EBD;

therefore the angles CBE, EBD are equal to the three angles CBA, ABE, EBD. [*Axiom* 2.

Again, the angle DBA is equal to the two angles DBE, EBA;

to each of these equals add the angle ABC;

therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC. [*Axiom* 2.

But the angles CBE, EBD have been shewn to be equal to the same three angles.

Therefore the angles CBE, EBD are equal to the angles DBA, ABC. [*Axiom* 1.

But CBE, EBD are two right angles;

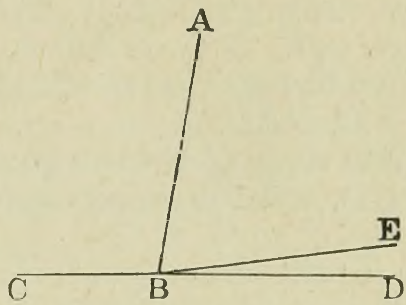
therefore DBA, ABC are together equal to two right angles.

Wherefore, *the angles* &c. Q.E.D.

PROPOSITION 14. *THEOREM.*

If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles: BD shall be in the same straight line with CB.



For if BD be not in the same straight line with CB, let BE be in the same straight line with it.

Then because the straight line AB makes with the straight line CBE, on one side of it, the angles ABC, ABE, these angles are together equal to two right angles. [I. 13.

But the angles ABC, ABD are also together equal to two right angles. [*Hypothesis.*

Therefore the angles ABC, ABE are equal to the angles ABC, ABD.

From each of these equals take away the common angle ABC, and the remaining angle ABE is equal to the remaining angle ABD, [*Axiom* 3.

the less to the greater; which is impossible.

Therefore BE is not in the same straight line with CB.

And in the same manner it may be shewn that no other can be in the same straight line with it but BD;

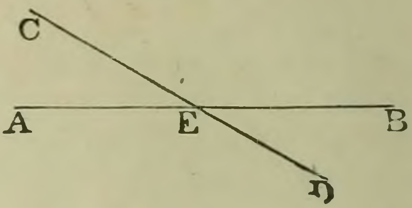
therefore BD is in the same straight line with CB.

Wherefore, *if at a point* &c. Q.E.D.

PROPOSITION 15. *THEOREM.*

If two straight lines cut one another, the vertical, or opposite, angles shall be equal.

Let the two straight lines AB, CD cut one another at the point E; the angle AEC shall be equal to the angle DEB, and the angle CEB to the angle AED.



Because the straight line AE makes with the straight line CD the angles CEA, AED, these angles are together equal to two right angles. [I. 13.
Again, because the straight line DE makes with the straight line AB the angles AED, DEB, these also are together equal to two right angles. [I. 13.
But the angles CEA, AED have been shewn to be together equal to two right angles.
Therefore the angles CEA, AED are equal to the angles AED, DEB.
From each of these equals take away the common angle AED, and the remaining angle CEA is equal to the remaining angle DEB. [*Axiom* 3.

In the same manner it may be shewn that the angle CEB is equal to the angle AED.

Wherefore, *if two straight lines* &c. Q.E.D.

Corollary 1. From this it is manifest that, if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.

Corollary 2. And consequently, that all the angles made by any number of straight lines meeting at one point, are together equal to four right angles.

PROPOSITION 16. *THEOREM.*

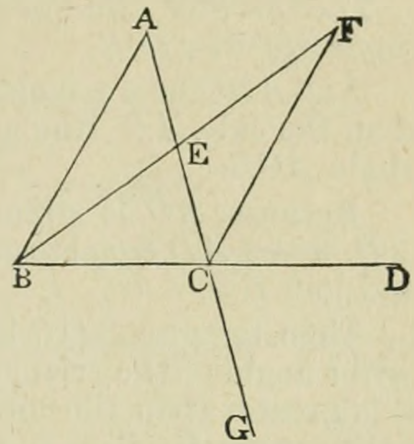
If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be a triangle, and let one side BC be produced to D: the exterior angle ACD shall be greater than either of the interior opposite angles CBA, BAC.

Bisect AC at E, [I. 10.
join BE and produce it to F, making EF equal to EB, [I. 3.
and join FC.

Because AE is equal to EC, and BE to EF; [*Constr.*
the two sides AE, EB are equal to the two sides CE, EF each to each;

and the angle AEB is equal to the angle CEF, because they are opposite vertical angles; [I. 15.



therefore the triangle AEB is equal to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; [I. 4.

therefore the angle BAE is equal to the angle ECF.

But the angle ECD is greater than the angle ECF. [*Axiom* 9.

Therefore the angle ACD is greater than the angle BAE.

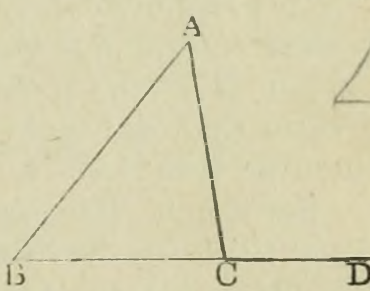
In the same manner if BC be bisected, and the side AC be produced to G, it may be shewn that the angle BCG, that is the angle ACD, is greater than the angle ABC. [I. 15.

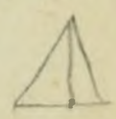
Wherefore, *if one side* &c. Q.E.D.

PROPOSITION 17. *THEOREM.*

Any two angles of a triangle are together less than two right angles.

Let ABC be a triangle: any two of its angles are together less than two right angles.





Produce BC to D.

Then because ACD is the exterior angle of the triangle ABC, it is greater than the interior opposite angle ABC. [I. 16.

To each of these add the angle ACB.

Therefore the angles ACD, ACB are greater than the angles ABC, ACB.

But the angles ACD, ACB are together equal to two right angles. [I. 13.

Therefore the angles ABC, ACB are together less than two right angles.

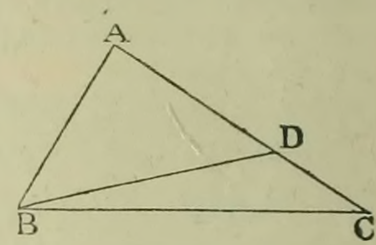
In the same manner it may be shewn that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles.

Wherefore, *any two angles* &c. Q.E.D.

PROPOSITION 18. *THEOREM.*

The greater side of every triangle has the greater angle opposite to it.

Let ABC be a triangle, of which the side AC is greater than the side AB: the angle ABC is also greater than the angle ACB.



Because AC is greater than AB, make AD equal to AB, [I. 3. and join BD.

Then, because ADB is the exterior angle of the triangle BDC, it is greater than the interior opposite angle DCB. [I. 16.

But the angle ADB is equal to the angle ABD, [I. 5.
because the side AD is equal to the side AB. [*Constr.*

Therefore the angle ABD is also greater than the angle ACB.

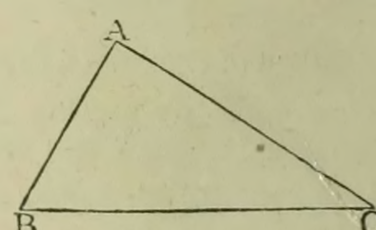
Much more then is the angle ABC greater than the angle ACB. [*Axiom* 9.

Wherefore, *the greater side* &c. Q.E.D.

PROPOSITION 19. *THEOREM.*

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let ABC be a triangle, of which the angle ABC is greater than the angle ACB: the side AC is also greater than the side AB.



For if not, AC must be either equal to AB or less than AB.

But AC is not equal to AB,
for then the angle ABC would be equal to the angle ACB; [I. 5.
but it is not; [*Hypothesis.*
therefore AC is not equal to AB.

Neither is AC less than AB,
for then the angle ABC would be less than the angle ACB; [I. 18.
but it is not; [*Hypothesis.*

therefore AC is not less than AB.

And it has been shewn that AC is not equal to AB.

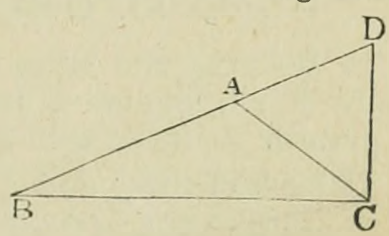
Therefore AC is greater than AB.

Wherefore, *the greater angle* &c. Q.E.D.

PROPOSITION 20. *THEOREM.*

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle: any two sides of it are together greater than the third side; namely, BA, AC greater than BC; and AB, BC greater than AC; and BC, CA greater than AB.



Produce BA to D, making AD equal to AC, [I. 3.
and join DC.

Then, because AD is equal to AC, [*Construction.*
the angle ADC is equal to the angle ACD. [I. 5.
But the angle BCD is greater than the angle ACD. [*Ax.* 9
Therefore the angle BCD is greater than the angle BDC.
And because the angle BCD of the triangle BCD is greater than its angle BDC, and that the greater angle is subtended by the greater side; [I. 19.
therefore the side BD is greater than the side BC.
But BD is equal to BA and AC.
Therefore BA, AC are greater than BC.

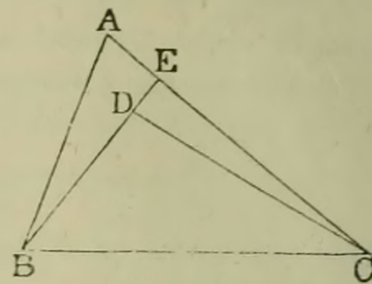
In the same manner it may be shewn that AB, BC are greater than AC, and BC, CA greater than AB.

Wherefore, *any two sides* &c. Q.E.D.

PROPOSITION 21. *THEOREM.*

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to the point D within the triangle: BD, DC shall be less than the other two sides BA, AC of the triangle, but shall contain an angle BDC greater than the angle BAC.



Produce BD to meet AC at E.

Because two sides of a triangle are greater than the third side, the two sides BA, AE of the triangle ABE are greater than the side BE. [I. 20.

To each of these add EC.

Therefore BA, AC are greater than BE, EC.

Again; the two sides CE, ED of the triangle CED are greater than the third side CD. [I. 20.

To each of these add DB.

Therefore CE, EB are greater than CD, DB.

But it has been shewn that BA, AC are greater than BE, EC;

much more then are BA, AC greater than BD, DC.

Again, because the exterior angle of any triangle is greater than the interior opposite angle, the exterior angle BDC of the triangle CDE is greater than the angle CED. [I. 16.

For the same reason, the exterior angle CEB of the triangle ABE is greater than the angle BAE.

But it has been shewn that the angle BDC is greater than the angle CEB;

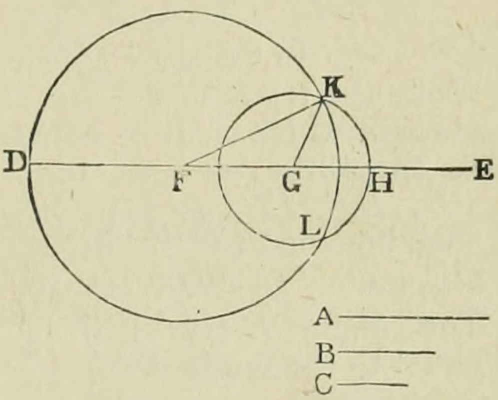
much more then is the angle BDC greater than the angle BAC.

Wherefore, *if from the ends* &c. Q.E.D.

PROPOSITION 22. *PROBLEM.*

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third; namely, A and B greater than C; A and C greater than B; and B and C greater than A: it is required to make a triangle of which the sides shall be equal to A, B, C, each to each.



Take a straight line DE terminated at the point D, but unlimited towards E, and make DF equal to A, FG equal to B, and GH equal to C. [I. 3.

From the centre F, at the distance FD, describe the circle DKL. [*Post.* 3.

From the centre G, at the distance GH, describe the circle HLK, cutting the former circle at K.

Join KF, KG. The triangle KFG shall have its sides equal to the three straight lines A, B, C.

Because the point F is the centre of the circle DKL, FD is equal to FK. [*Definition* 15.

But FD is equal to A. [*Construction.*

Therefore FK is equal to A. [*Axiom* 1.

Again, because the point G is the centre of the circle HLK, GH is equal to GK. [*Definition* 15.

But GH is equal to C. [*Construction.*

Therefore GK is equal to C. [*Axiom* 1.

And FG is equal to B. [*Construction.*

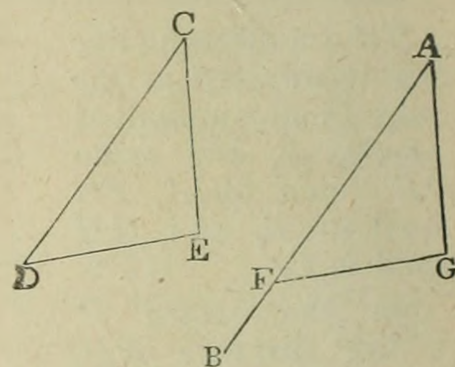
Therefore the three straight lines KF, FG, GK are equal to the three A, B, C.

Wherefore *the triangle* KFG *has its three sides* KF, FG, GK *equal to the three given straight lines* A, B, C. Q.E.F.

PROPOSITION 23. *PROBLEM.*

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle: it is required to make at the given point A, in the given straight line AB, an angle equal to the given rectilineal angle DCE.



In CD, CE take any points D, E, and join DE. Make the triangle AFG the sides of which shall be equal to the three straight lines CD, DE, EC; so that AF shall be equal to CD, AG to CE, and FG to DE. [I. 22.

The angle FAG shall be equal to the angle DCE.

Because FA, AG are equal to DC, CE, each to each, and the base FG equal to the base DE; [*Construction.*

therefore the angle FAG is equal to the angle DCE. [I. 8.

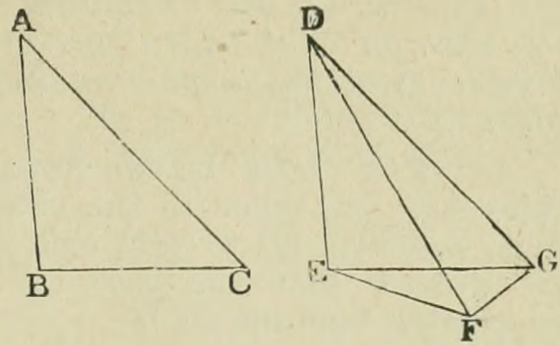
Wherefore *at the given point A in the given straight line AB, the angle FAG has been made equal to the given rectilineal angle DCE.* Q.E.F.

PROPOSITION 24. *THEOREM.*

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC, equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF, but the angle BAC greater than the angle EDF: the base BC shall be

greater than the base *EF*.



Of the two sides *DE*, *DF*, let *DE* be the side which is not greater than the other. At the point *D* in the straight line *DE*, make the angle *EDG* equal to the angle *BAC*, [I. 23.

and make *DG* equal to *AC* or *DF*, [I. 3.
and join *EG*, *GF*.

Because *AB* is equal to *DE*, [*Hypothesis.*
and *AC* to *DG*; [*Construction.*
the two sides *BA*, *AC* are equal to the two sides *ED*, *DG*, each to each;
and the angle *BAC* is equal to the angle *EDG*; [*Constr.*
therefore the base *BC* is equal to the base *EG*. [I. 4.

And because *DG* is equal to *DF*, [*Construction.*
the angle *DGF* is equal to the angle *DFG*. [I. 5.
But the angle *DGF* is greater than the angle *EGF*. [*Ax.* 9.
Therefore the angle *DFG* is greater than the angle *EGF*.
Much more then is the angle *EFG* greater than the angle *EGF*. [*Axiom* 9.

And because the angle *EFG* of the triangle *EFG* is greater than its angle *EGF*, and that the greater angle is subtended by the greater side, [I. 19.
therefore the side *EG* is greater than the side *EF*.
But *EG* was shewn to be equal to *BC*;
therefore *BC* is greater than *EF*.

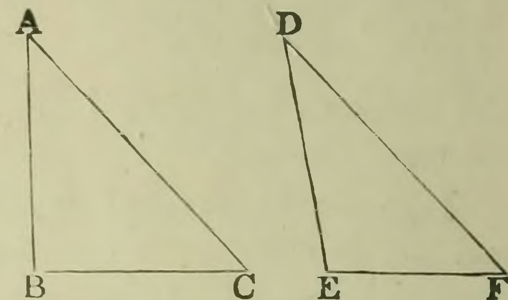
Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 25. *THEOREM.*

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one

greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them, of the other.

Let *ABC*, *DEF* be two triangles, which have the two sides *AB*, *AC* equal to the two sides *DE*, *DF*, each to each, namely, *AB* to *DE*, and *AC* to *DF*, but the base *BC* greater than the base *EF*: the angle *BAC* shall be greater than the angle *EDF*.



For if not, the angle *BAC* must be either equal to the angle *EDF* or less than the angle *EDF*.

But the angle *BAC* is not equal to the angle *EDF*, for then the base *BC* would be equal to the base *EF*; [I. 4.

but it is not; [*Hypothesis.*

therefore the angle *BAC* is not equal to the angle *EDF*.

Neither is the angle *BAC* less than the angle *EDF*,

for then the base *BC* would be less than the base *EF*; [I. 24.

but it is not; [*Hypothesis.*

therefore the angle *BAC* is not less than the angle *EDF*.

And it has been shewn that the angle *BAC* is not equal to the angle *EDF*.

Therefore the angle *BAC* is greater than the angle *EDF*.

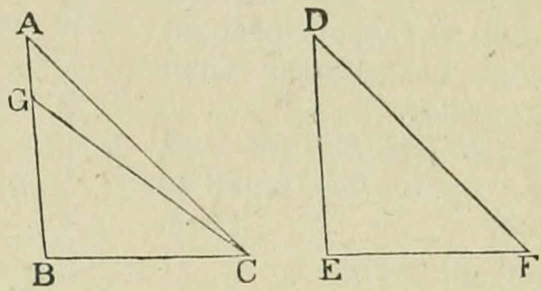
Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 26. *THEOREM.*

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

Let *ABC*, *DEF* be two triangles, which have the angles *ABC*, *BCA* equal to the angles *DEF*, *EFD*, each

to each, namely, ABC to DEF, and BCA to EFD; and let them have also one side equal to one side; and first let those sides be equal which are adjacent to the equal angles in the two triangles, namely, BC to EF: the other sides shall be equal, each to each, namely, AB to DE, and AC to DF, and the third angle BAC equal to the third angle EDF.



For if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater, and make BG equal to DE, [I. 3.

and join GC.

Then because GB is equal to DE, [*Construction.*

and BC to EF; [*Hypothesis.*

the two sides GB, BC are equal to the two sides DE, EF, each to each;

and the angle GBC is equal to the angle DEF; [*Hypothesis.*

therefore the triangle GBC is equal to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.

therefore the angle GCB is equal to the angle DFE.

But the angle DFE is equal to the angle ACB. [*Hypothesis.*

Therefore the angle GCB is equal to the angle ACB, [*Ax.* 1.

the less to the greater; which is impossible.

Therefore AB is not unequal to DE,

that is, it is equal to it;

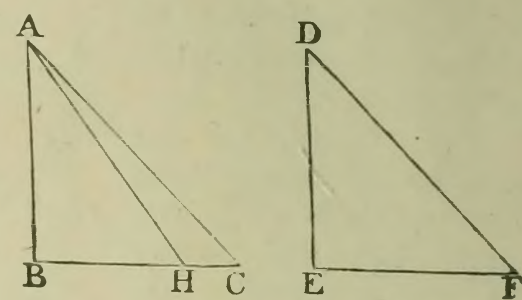
and BC is equal to EF; [*Hypothesis.*

therefore the two sides AB, BC are equal to the two sides DE, EF, each to each;

and the angle ABC is equal to the angle DEF; [*Hypothesis.*

therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF. [I. 4.

Next, let sides which are opposite to equal angles in each triangle be equal to one another, namely, AB to DE: likewise in this case the other sides shall be equal, each to each, namely, BC to EF, and AC to DF, and also the third angle BAC equal to the third angle EDF.



For if BC be not equal to EF, one of them must be greater than the other. Let BC be the greater, and make BH equal to EF, [I. 3.
and join AH.

Then because BH is equal to EF, [*Construction.*
and AB to DE; [*Hypothesis.*
the two sides AB, BH are equal to the two sides DE, EF, each to each;
and the angle ABH is equal to the angle DEF; [*Hypothesis.*
therefore the triangle ABH is equal to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.
therefore the angle BHA is equal to the angle EFD.
But the angle EFD is equal to the angle BCA. [*Hypothesis.*
Therefore the angle BHA is equal to the angle BCA; [*Ax.* 1.
that is, the exterior angle BHA of the triangle AHC is equal to its interior opposite angle BCA;
which is impossible. [I. 16.
Therefore BC is not unequal to EF,
that is, it is equal to it;
and AB is equal to DE; [*Hypothesis.*
therefore the two sides AB, BC are equal to the two sides DE, EF, each to each;
and the angle ABC is equal to the angle DEF; [*Hypothesis.*
therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF. [I. 4.

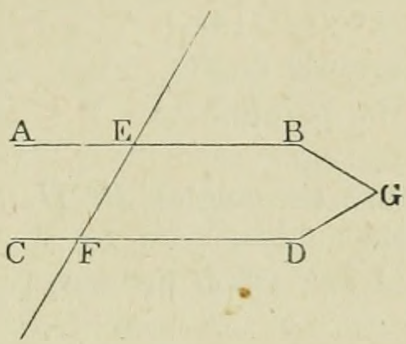
Wherefore, *if two triangles* &c. Q.E.D.

PROPOSITION 27. *THEOREM.*

If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

Let the straight line *EF*, which falls on the two straight lines *AB*, *CD*, make the alternate angles *AEF*, *EFD* equal to one another: *AB* shall be parallel to *CD*.

For if not, *AB* and *CD*, being produced, will meet either towards *B*, *D* or towards *A*, *C*. Let them be produced and meet towards *B*, *D* at the point *G*.



Therefore *GEF* is a triangle, and its exterior angle *AEF* is greater than the interior opposite angle *EFG*; [I. 16.

But the angle *AEF* is also equal to the angle *EFG*; [*Hyp.* which is impossible.

Therefore *AB* and *CD* being produced, do not meet towards *B*, *D*.

In the same manner, it may be shewn that they do not meet towards *A*, *C*.

But those straight lines which being produced ever so far both ways do not meet, are parallel. [*Definition* 35.

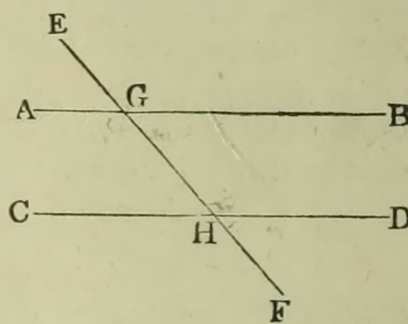
Therefore *AB* is parallel to *CD*.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 28. *THEOREM.*

If a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line *EF*, which falls on the two straight lines *AB*, *CD*, make the exterior angle *EGB* equal to the interior and opposite angle *GHD* on the same side, or make the interior angles on the same side *BGH*, *GHD* together equal to two right angles: *AB* shall be parallel to *CD*.



Because the angle *EGB* is equal to the angle *GHD*, [*Hyp.*
and the angle *EGB* is also equal to the angle *AGH*, [I. 15.
therefore the angle *AGH* is equal to the angle *GHD*; [*Ax.*1.
and they are alternate angles; therefore *AB* is parallel to *CD*. [I. 27.

Again; because the angles *BGH*, *GHD* are together equal to two right angles, [*Hypothesis.*
and the angles *AGH*, *BGH* are also together equal to two right angles, [I. 13.
therefore the angles *AGH*, *BGH* are equal to the angles *BGH*, *GHD*.
Take away the common angle *BGH*; therefore the remaining angle *AGH* is equal to the remaining angle *GHD*; [*Axiom* 3.
and they are alternate angles;
therefore *AB* is parallel to *CD*. [I. 27.

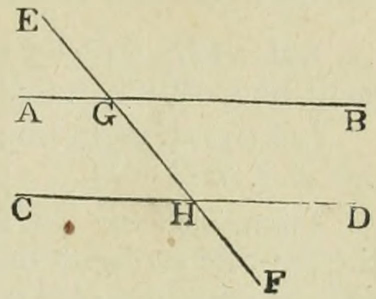
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 29. *THEOREM.*

If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

Let the straight line *EF* fall on the two parallel straight lines *AB*, *CD*: the alternate angles *AGH*, *GHD* shall be equal to one another, and the exterior angle *EGB* shall be equal to the interior and opposite angle

on the same side, *GHD*, and the two interior angles on the same side, *BGH*, *GHD*, shall be together equal to two right angles.



For if the angle *AGH* be not equal to the angle *GHD*, one of them must be greater than the other; let the angle *AGH* be the greater.

Then the angle *AGH* is greater than the angle *GHD*;

to each of them add the angle *BGH*;

therefore the angles *AGH*, *BGH* are greater than the angles *BGH*, *GHD*.

But the angles *AGH*, *BGH* are together equal to two right angles; [I. 13.

therefore the angles *BGH*, *GHD* are together less than two right angles.

But if a straight line meet two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet on that side on which are the angles which are less than two right angles. [*Axiom* 12.

Therefore the straight lines *AB*, *CD*, if continually produced, will meet.

But they never meet, since they are parallel by hypothesis.

Therefore the angle *AGH* is not unequal to the angle *GHD*; that is, it is equal to it.

But the angle *AGH* is equal to the angle *EGB*. [I. 15.

Therefore the angle *EGB* is equal to the angle *GHD*. [*Ax.* 1.

Add to each of these the angle *BGH*.

Therefore the angles *EGB*, *BGH* are equal to the angles *BGH*, *GHD*. [*Axiom* 2.

But the angles *EGB*, *BGH* are together equal to two right angles. [I. 13.

Therefore the angles *BGH*, *GHD* are together equal to two right angles. [*Axiom* 1.

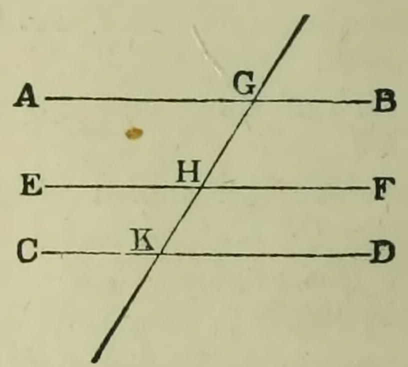
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 30. *THEOREM.*

Straight lines which are parallel to the same straight line are parallel to each other.

Let *AB*, *CD* be each of them parallel to *EF*: *AB* shall be parallel to *CD*.

Let the straight line *GHK* cut *AB*, *EF*, *CD*.



Then, because *GHK* cuts the parallel straight lines *AB*, *EF*, the angle *AGH* is equal to the angle *GHF*. [I. 29.

Again, because *GK* cuts the parallel straight lines *EF*, *CD*, the angle *GHF* is equal to the angle *GKD*. [I. 29.

And it was shewn that the angle *AGK* is equal to the angle *GHF*.

Therefore the angle *AGK* is equal to the angle *GKD*; [*Ax.* 1.

and they are alternate angles;

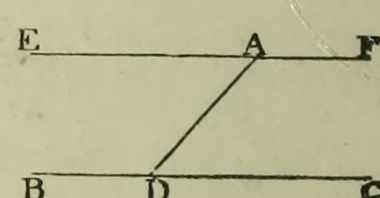
therefore *AB* is parallel to *CD*. [I. 27.

Wherefore, *straight lines* &c. Q.E.D.

PROPOSITION 31. *PROBLEM.*

To draw a straight line through a given point parallel to a given straight line.

Let *A* be the given point, and *BC* the given straight line: it is required to draw a straight line through the point *A* parallel to the straight line *BC*.



In *BC* take any point *D*, and join *AD*; at the point *A* in the straight line *AD*, make the angle *DAE* equal to the angle *ADC*; [I. 23.

and produce the straight line *EA* to *F*.

EF shall be parallel to *BC*.

Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another, [*Construction.*

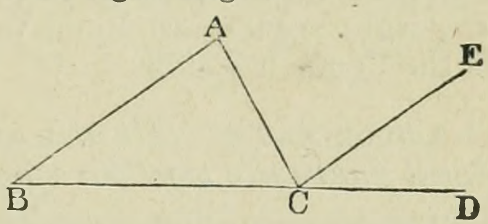
EF is parallel to BC. [I. 27.

Wherefore *the straight line* EAF *is drawn through the given point* A, *parallel to the given straight line* BC. Q.E.F.

PROPOSITION 32. *THEOREM.*

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D: the exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC; and the three interior angles of the triangle, namely, ABC, BCA, CAB shall be equal to two right angles.



Through the point C draw CE parallel to AB. [I. 31.

Then, because AB is parallel to CE, and AC falls on them, the alternate angles BAC, ACE are equal. [I. 29.

Again, because AB is parallel to CE, and BD falls on them, the exterior angle ECD is equal to the interior and opposite angle ABC. [I. 29.

But the angle ACE was shewn to be equal to the angle BAC;

therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC. [*Axiom* 2.

To each of these equals add the angle ACB;

therefore the angles ACD, ACB are equal to the three angles CBA, BAC, ACB. [*Axiom* 2.

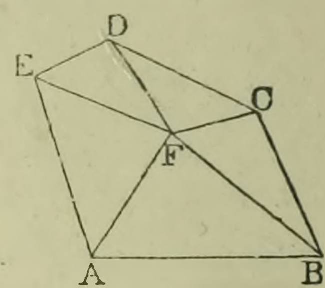
But the angles ACD, ACB are together equal to two right angles; [I. 13.

therefore also the angles CBA, BAC, ACB are together equal to two right angles. [*Axiom* 1.

Wherefore, *if a side of any triangle* &c. Q.E.D.

Corollary 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

For any rectilineal figure $ABCDE$ can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles.

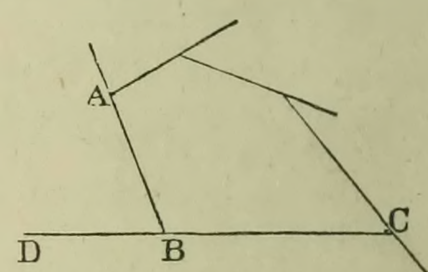


And by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as the figure has sides.

And the same angles are equal to the interior angles of the figure, together with the angles at the point F, which is the common vertex of the triangles, that is, together with four right angles. [I. 15. *Corollary* 2.

Therefore all the interior angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Corollary 2. *All the exterior angles of any rectilineal figure are together equal to four right angles.*



Because every interior angle ABC, with its adjacent exterior angle ABD, is equal to two right angles; [I. 13.

therefore all the interior angles of the figure, together with all its exterior angles, are equal to twice as many right angles as the figure has sides.

But, by the foregoing Corollary all the interior angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Therefore all the interior angles of the figure, together with all its exterior angles, are equal to all the interior angles of the figure, together with four right angles.

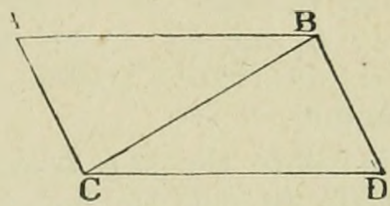
Therefore all the exterior angles are equal to four right angles.

PROPOSITION 33. *THEOREM.*

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let AB and CD be equal and parallel straight lines, and let them be joined towards the same parts by the straight lines AC and BD: AC and BD shall be equal and parallel.

Join BC.



Then because AB is parallel to CD, [*Hypothesis.*
and BC meets them,
the alternate angles ABC, BCD are equal. [I. 29.

And because AB is equal to CD, [*Hypothesis.*
and BC is common to the two triangles ABC, DCB;
the two sides AB, BC are equal to the two sides DC, CB, each to each;
and the angle ABC was shewn to be equal to the angle BCD;
therefore the base AC is equal to the base BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.
therefore the angle ACB is equal to the angle CBD.

And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another, AC is parallel to BD. [I. 27.
And it was shewn to be equal to it.

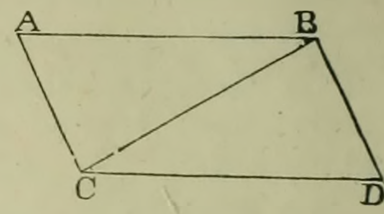
Wherefore, *the straight lines* &c. Q.E.D.

PROPOSITION 34. *THEOREM.*

The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram, that is, divides it into two equal parts.

Note. A parallelogram is a four-sided figure of which the opposite sides are parallel; and a diameter is the straight line joining two of its opposite angles.

Let $ACDB$ be a parallelogram, of which BC is a diameter; the opposite sides and angles of the figure shall be equal to one another, and the diameter BC shall bisect it.



Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal to one another. [I. 29.

And because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal to one another. [I. 29.

Therefore the two triangles ABC, BCD have two angles ABC, BCA in the one, equal to two angles DCB, CBD in the other, each to each, and one side BC is common to the two triangles, which is adjacent to their equal angles;

therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other, namely, the side AB equal to the side CD, and the side AC equal to the side BD, and the angle BAC equal to the angle CDB [I. 26.

And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB,

the whole angle ABD is equal to the whole angle ACD. [*Ax.* 2.

And the angle BAC has been shewn to be equal to the angle CDB.

Therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diameter bisects the parallelogram.

For AB being equal to CD, and BC common,

the two sides AB, BC are equal to the two sides DC, CB each to each;

and the angle ABC has been shewn to be equal to the angle BCD;

therefore the triangle ABC is equal to the triangle BCD, [I. 4.

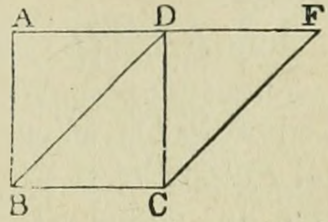
and the diameter BC divides the parallelogram $ACDB$ into two equal parts.

Wherefore, *the opposite sides* &c. Q.E.D.

PROPOSITION 35. *THEOREM.*

Parallelograms on the same base, and between the same parallels, are equal to one another.

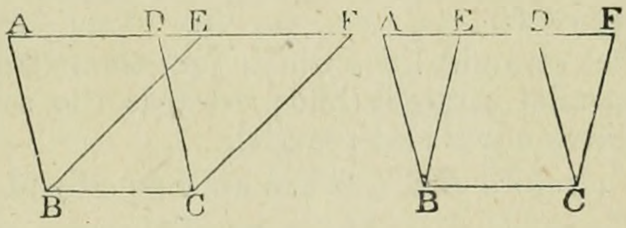
Let the parallelograms $ABCD$, $EBCF$ be on the same base BC, and between the same parallels AF, BC: the parallelogram $ABCD$ shall be equal to the parallelogram $EBCF$.



If the sides AD, DF of the parallelograms $ABCD$, $DBCF$, opposite to the base BC, be terminated at the same point D, it is plain that each of the parallelograms is double of the triangle BDC; [I. 34.

and they are therefore equal to one another. [*Axiom* 6.

But if the sides AD, EF, opposite to the base BC of the parallelograms $ABCD$, $EBCF$ be not terminated at the same point, then, because $ABCD$ is a parallelogram AD is equal to BC; [I. 34.



for the same reason EF is equal to BC;

therefore AD is equal to EF; [*Axiom* 1.

therefore the whole, or the remainder, AE is equal to the whole, or the remainder, DF. [*Axioms* 2, 3.

And AB is equal to DC; [I. 34.

therefore the two sides EA, AB are equal to the two sides FD, DC each to each;

and the exterior angle FDC is equal to the interior and opposite angle EAB; [I. 29.

therefore the triangle EAB is equal to the triangle FDC. [I. 4.

Take the triangle FDC from the trapezium $ABCF$, and from [redacted] a smaller trapezium take the triangle EAB, and the remainders are equal; [*Axiom* 3.

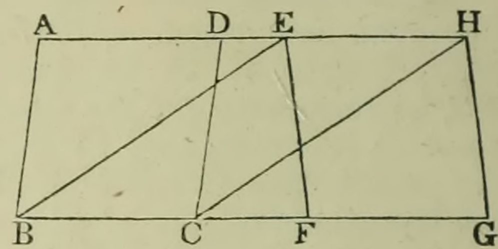
that is, the parallelogram $ABCD$ is equal to the parallelogram $EBCF$.

Wherefore, *parallelograms on the same base* &c. Q.E.D.

PROPOSITION 36. *THEOREM.*

Parallelograms on equal bases, and between the same parallels, are equal to one another.

Let *ABCD*, *EFGH* be parallelograms on equal bases *BC*, *FG*, and between the same parallels *AH*, *BG*: the parallelogram *ABCD* shall be equal to the parallelogram *EFGH*.



Join *BE*, *CH*.

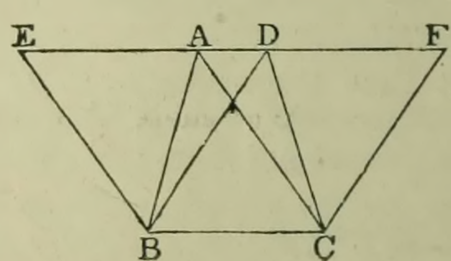
Then, because *BC* is equal to *FG*, [*Hyp.*
and *FG* to *EH*, [I. 34.
BC is equal to *EH*; [*Axiom* 1.
and they are parallels, [*Hypothesis.*
and joined towards the same parts by the straight lines *BE*, *CH*.
But straight lines which join the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel. [I. 33.
Therefore *BE*, *CH* are both equal and parallel.
Therefore *EBCH* is a parallelogram. [*Definition.*
And it is equal to *ABCD*, because they are on the same base *BC*, and between the same parallels *BC*, *AH*. [I. 35.

For the same reason the parallelogram *EFGH* is equal to the same *EBCH*.
Therefore the parallelogram *ABCD* is equal to the parallelogram *EFGH*. [*Axiom* 1.

Wherefore, *parallelograms* &c. Q.E.D.

PROPOSITION 37. *THEOREM.*

Triangles on the same base, and between the same parallels, are equal.



Let the triangles *ABC*, *DBC* be on the same base *BC*, and between the same parallels *AD*, *BC*: the triangle *ABC* shall be equal to the triangle *DBC*.

Produce *AD* both ways to the points *E*, *F*; [*Post.* 2.

through B draw BE parallel to CA, and through C draw CF parallel to BD. [I. 31.

Then each of the figures $EBCA$, $DBCF$ is a parallelogram; [*Definition.*

and $EBCA$ is equal to $DBCF$, because they are on the same base BC, and between the same parallels BC, EF. [I. 35.

And the triangle ABC is half of the parallelogram $EBCA$, because the diameter AB bisects the parallelogram; [I. 34.

and the triangle DBC is half of the parallelogram $DBCF$, because the diameter DC bisects the parallelogram. [I. 34.

But the halves of equal things are equal. [*Axiom* 7.

Therefore the triangle ABC is equal to the triangle DBC.

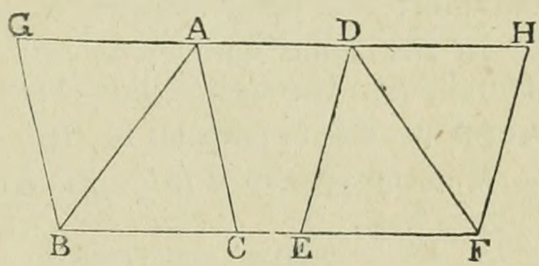
Wherefore, *triangles* &c. Q.E.D.

PROPOSITION 38. *THEOREM.*

Triangles on equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD: the triangle ABC shall be equal to the triangle DEF.

Produce AD both ways to the points G, H; through B draw BG parallel to CA, and through F draw FH parallel to ED. [I. 31.



Then each of the figures $GBCA$, $DEFH$ is a parallelogram. [*Definition.*

And they are equal to one another because they are on equal bases BC, EF, and between the same parallels BF, GH. [I. 36.

And the triangle ABC is half of the parallelogram $GBCA$, because the diameter AB bisects the parallelogram; [I. 34.

and the triangle DEF is half of the parallelogram $DEFH$, because the diameter DF bisects the parallelogram.

But the halves of equal things are equal. [*Axiom* 7.

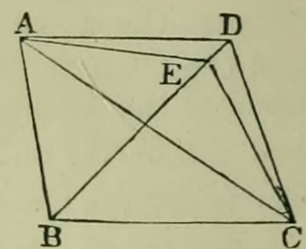
Therefore the triangle ABC is equal to the triangle DEF.

Wherefore, *triangles* &c. Q.E.D.

PROPOSITION 39. *THEOREM.*

Equal triangles on the same base, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be on the same base BC, and on the same side of it: they shall be between the same parallels.



Join AD.

AD shall be parallel to BC.

For if it is not, through A draw AE parallel to BC, meeting BD at E. [I. 31.

and join EC.

Then the triangle ABC is equal to the triangle EBC, because they are on the same base BC, and between the same parallels BC, AE. [I. 37.

But the triangle ABC is equal to the triangle DBC. [*Hyp.*

Therefore also the triangle DBC is equal to the triangle EBC, [*Axiom* 1.

the greater to the less; which is impossible.

Therefore AE is not parallel to BC.

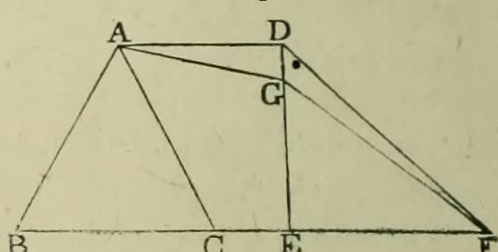
In the same manner it can be shewn, that no other straight line through A but AD is parallel to BC; therefore AD is parallel to BC.

Wherefore, *equal triangles* &c. Q.E.D.

PROPOSITION 40. *THEOREM.*

Equal triangles, on equal bases, in the same straight line, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DEF be on equal bases BC, EF, in the same straight line BF, and on the same side of it: they shall be between the same parallels.



Join AD.

AD shall be parallel to BF.

For if it is not, through A draw AG parallel to BF, meeting ED at G [I. 31.

and join GF.

Then the triangle ABC is equal to the triangle GEF, because they are on equal bases BC, EF, and between the same parallels. [I. 38.

But the triangle ABC is equal to the triangle DEF. [*Hyp.*

Therefore also the triangle DEF is equal to the triangle GEF, [*Axiom* 1.

the greater to the less; which is impossible.

Therefore AG is not parallel to BF.

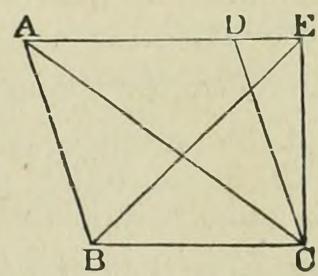
In the same manner it can be shewn that no other straight line through A but AD is parallel to BF; therefore AD is parallel to BF.

Wherefore, *equal triangles* &c. Q.E.D.

PROPOSITION 41. *THEOREM.*

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram $ABCD$ and the triangle EBC be on the same base BC, and between the same parallels BC, AE: the parallelogram $ABCD$ shall be double of the triangle EBC.



Join AC.

Then the triangle ABC is equal to the triangle EBC, because they are on the same base BC, and between the same parallels BC, AE. [I. 37.

But the parallelogram $ABCD$ is double of the triangle ABC, because the diameter AC bisects the parallelogram. [I. 34.

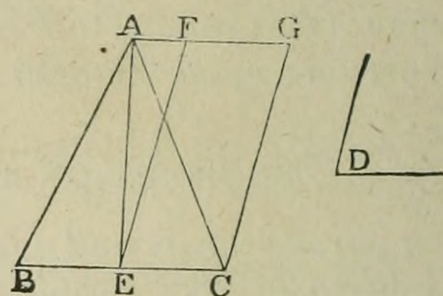
Therefore the parallelogram $ABCD$ is also double of the triangle EBC.

Wherefore, *if a parallelogram* &c. Q.E.D.

PROPOSITION 42. *PROBLEM.*

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle: it is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



Bisect BC at E: [I. 10.
join AE, and at the point E, in the straight line EC, make the angle CEF equal to D; [I. 23.
through A draw AFG parallel to EC, and through C draw CG parallel to EF. [I. 31.

Therefore $FECG$ is a parallelogram. [*Definition.*

And, because BE is equal to EC, [*Construction.*

the triangle ABE is equal to the triangle AEC, because they are on equal bases BE, EC, and between the same parallels BC, AG. [I. 38.

Therefore the triangle ABC is double of the triangle AEC.

But the parallelogram $FECG$ is also double of the triangle AEC, because they are on the same base EC, and between the same parallels EC, AG. [I. 41.

Therefore the parallelogram $FECG$ is equal to the triangle ABC; [*Axiom* 6.

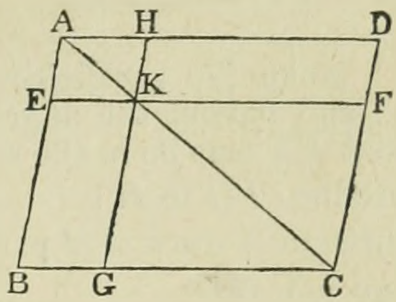
and it has one of its angles CEF equal to the given angle D. [*Construction.*

Wherefore *a parallelogram $FECG$ has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D.* Q.E.F.

PROPOSITION 43. *THEOREM.*

The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let *ABCD* be a parallelogram, of which the diameter is *AC*; and *EH*, *GF* parallelograms about *AC*, that is. through which *AC* passes; and *BK*, *KD* the other parallelograms which make up the whole figure *ABCD*, and which are therefore called the complements: the complement *BK* shall be equal to the complement *KD*.



Because *ABCD* is a parallelogram, and *AC* its diameter, the triangle *ABC* is equal to the triangle *ADC*. [I. 34.

Again, because *AEKH* is a parallelogram, and *AK* its diameter, the triangle *AEK* is equal to the triangle *AHK*. [I. 34.

For the same reason the triangle *KGC* is equal to the triangle *KFC*.

Therefore, because the triangle *AEK* is equal to the triangle *AHK*, and the triangle *KGC* to the triangle *KFC*; the triangle *AEK* together with the triangle *KGC* is equal to the triangle *AHK* together with the triangle *KFC*. [*Ax.* 2.

But the whole triangle *ABC* was shewn to be equal to the whole triangle *ADC*.

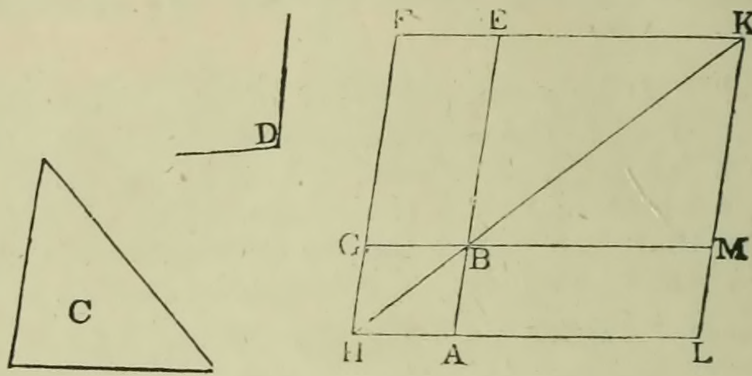
Therefore the remainder, the complement *BK*, is equal to the remainder, the complement *KD*. [*Axiom* 3.

Wherefore, *the complements* &c. Q.E.D.

PROPOSITION 44. *PROBLEM.*

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle: it is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.



Make the parallelogram $BEFG$ equal to the triangle C, and having the angle EBG equal to the angle D, so that BE may be in the same straight line with AB; [I. 42.
produce FG to H;
through A draw AH parallel to BG or EF, [I. 31.
and join HB.

Then, because the straight line HF falls on the parallels AH, EF, the angles AHF, HFE are together equal to two right angles. [I. 29.
Therefore the angles BHF, HFE are together less than two right angles.
But straight lines which with another straight line make the interior angles on the same side together less than two right angles will meet on that side, if produced far enough. [*Ax.* 12.
Therefore HB and FE will meet if produced;
let them meet at K.
Through K draw KL parallel to EA or FH; [I. 31.
and produce HA, GB to the points L, M.

Then $HLKF$ is a parallelogram, of which the diameter is HK; and AG, ME are parallelograms about HK; and LB, BF are the complements.
Therefore LB is equal to BF. [I. 43.
But BF is equal to the triangle C. [*Construction.*
Therefore LB is equal to the triangle C. [*Axiom* 1.

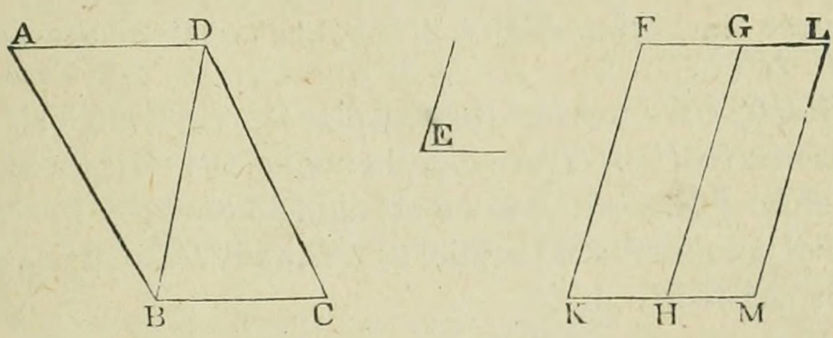
And because the angle GBE is equal to the angle ABM, [I.15.
and likewise to the angle D; [*Construction.*
the angle ABM is equal to the angle D. [*Axiom* 1.

Wherefore *to the given straight line AB the parallelogram LB is applied, equal to the triangle C, and having the angle ABM equal to the angle D.* Q.E.F.

PROPOSITION 45. *PROBLEM.*

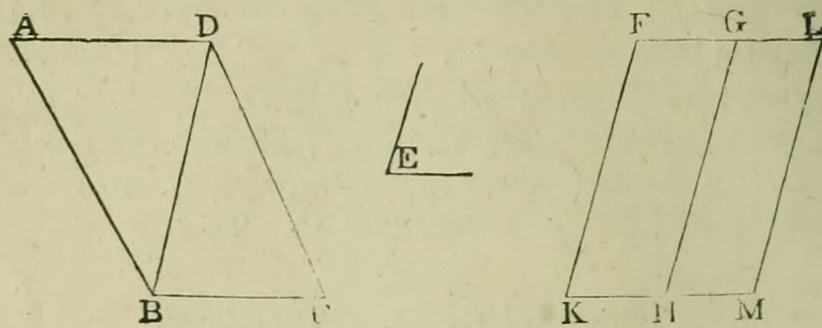
To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let $ABCD$ be the given rectilineal figure, and E the given rectilineal angle: it is required to describe a parallelogram equal to $ABCD$, and having an angle equal to E.



Join DB, and describe the parallelogram FH equal to the triangle ADB, and having the angle FKH equal to the angle E; [I. 42.
and to the straight line GH apply the parallelogram GM equal to the triangle DBC, and having the angle GHM equal to the angle E. [I. 44.
The figure $FKML$ shall be the parallelogram required.

Because the angle E is equal to each of the angles FKH, GHM, [*Construction.*
the angle FKH is equal to the angle GHM. [*Axiom* 1.
Add to each of these equals the angle KHG;
therefore the angles FKH, KHG are equal to the angles KHG, GHM. [*Axiom* 2.
But FKH, KHG are together equal to two right angles; [I.29.
therefore KHG, GHM are together equal to two right angles.



And because at the point H in the straight line GH, the two straight lines KH, HM, on the opposite sides of it, make the adjacent angles together equal to two right angles, KH is in the same straight line with HM. [I. 14.

And because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal. [I. 29.
Add to each of these equals the angle HGL;
therefore the angles MHG, HGL, are equal to the angles HGF, HGL. [*Axiom* 2.
But MHG, HGL are together equal to two right angles; [I. 29.
therefore HGF, HGL are together equal to two right angles
Therefore FG is in the same straight line with GL. [I. 14

And because KF is parallel to HG, and HG to ML, [*Constr.*
KF is parallel to ML; [I. 30.
and KM, FL are parallels; [*Construction.*
therefore $KFLM$ is a parallelogram. [*Definition.*
And because the triangle ABD is equal to the parallelogram HF, [*Construction.*
and the triangle DBC to the parallelogram GM; [*Constr.*
the whole rectilineal figure $ABCD$ is equal to the whole parallelogram $KFLM$. [*Axiom* 2

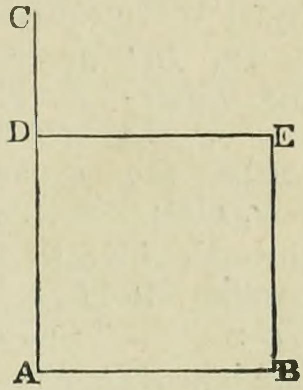
Wherefore, *the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E.* Q.E.F.

Corollary. From this it is manifest, how to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle; and so on. [I. 44

PROPOSITION 46. *PROBLEM.*

To describe a square on a given straight line.

Let AB be the given straight line: it is required to describe a square on AB.



From the point A draw AC at right angles to AB; [I. 11.
and make AD equal to AB; [I. 3.
through D draw DE parallel to AB; and through B draw BE parallel to AD. [I. 31.
$ADEB$ shall be a square.

For $ADEB$ is by construction a parallelogram;
therefore AB is equal to DE, and AD to BE. [I. 34.
But AB is equal to AD. [*Construction.*
Therefore the four straight lines BA, AD, DE, EB are equal to one another, and the parallelogram $ADEB$ is equilateral. [*Axiom* 1.

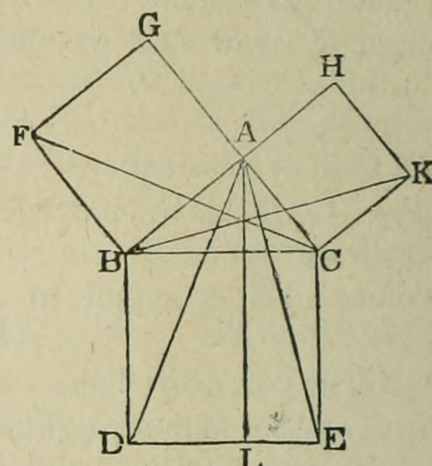
Likewise all its angles are right angles.
For since the straight line AD meets the parallels AB, DE, the angles BAD, ADE are together equal to two right angles; [I. 29.
but BAD is a right angle; [*Construction.*
therefore also ADE is a right angle. [*Axiom* 3.
But the opposite angles of parallelograms are equal. [I. 34.
Therefore each of the opposite angles ABE, BED is a right angle. [*Axiom* 1.
Therefore the figure $ADEB$ is rectangular;
and it has been shewn to be equilateral.
Therefore *it is a square.* [*Definition* 30.
And it is described on the given straight line AB. Q.E.F.

Corollary. From the demonstration it is manifest that every parallelogram which has one right angle has all its angles right angles.

PROPOSITION 47. *THEOREM.*

In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC: the square described on the side BC shall be equal to the squares described on the sides BA, AC.



On BC describe the square $BDEC$, and on BA, AC describe the squares GB, HC; [I. 46.

through A draw AL parallel to BD or CE; [I. 31.

and join AD, FC.

Then, because the angle BAC is a right angle, [*Hypothesis.*

and that the angle BAG is also a right angle, [*Definition* 30.

the two straight lines AC, AG, on the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles;

therefore CA is in the same straight line with AG. [I. 14.

For the same reason, AB and AH are in the same straight line.

Now the angle DBC is equal to the angle FBA, for each of them is a right angle. [*Axiom* 11.

Add to each the angle ABC.

Therefore the whole angle DBA is equal to the whole angle FBC. [*Axiom* 2.

And because the two sides AB, BD are equal to the two sides FB, BC, each to each; [*Definition* 30.

and the angle DBA is equal to the angle FBC;

therefore the triangle ABD is equal to the triangle FBC. [I. 4.

Now the parallelogram BL is double of the triangle ABD, because they are on the same base BD, and between the same parallels BD, AL. [I. 41.

And the square GB is double of the triangle FBC, because they are on the same base FB, and between the same parallels FB, GC. [I. 41.

But the doubles of equals are equal to one another. [*Ax.* 6.

Therefore the parallelogram BL is equal to the square GB.

In the same manner, by joining AE, BK, it can be shewn, that the parallelogram CL is equal to the square CH.

Therefore the whole square $BDEC$ is equal to the two squares GB, HC. [*Axiom* 2.

And the square $BDEC$ is described on BC, and the squares GB, HC on BA, AC.

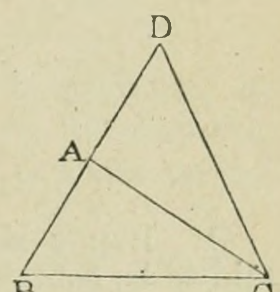
Therefore the square described on the side BC is equal to the squares described on the sides BA, AC.

Wherefore, *in any right-angled triangle* &c. Q.E.D.

PROPOSITION 48. *THEOREM.*

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described on BC, one of the sides of the triangle ABC, be equal to the squares described on the other sides BA, AC: the angle BAC shall be a right angle.



From the point A draw AD at right angles to AC; [I. 11.

and make AD equal to BA; [I. 3.

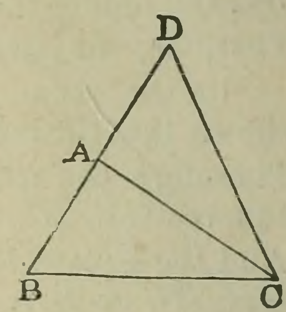
and join DC.

Then because DA is equal to BA, the square on DA is equal to the square on BA.

To each of these add the square on AC.

Therefore the squares on DA, AC are equal to the squares on BA, AC. [*Axiom* 2.

But because the angle DAC is a right angle, [*Construction.*
the square on DC is equal to the squares on DA, AC. [I. 47.
And, by hypothesis, the square on BC is equal to the squares on BA, AC.
Therefore the square on DC is equal to the square on BC. [*Ax.* 1.
Therefore also the side DC is equal to the side BC.



And because the side DA is equal to the side AB; [*Constr.*
and the side AC is common to the two triangles DAC, BAC;
the two sides DA, AC are equal to the two sides BA, AC, each to each;
and the base DC has been shewn to be equal to the base BC;
therefore the angle DAC is equal to the angle BAC. [I. 8.
But DAC is a right angle; [*Construction.*
therefore also BAC is a right angle. [*Axiom* 1.

Wherefore, *if the square* &c. Q.E.D.

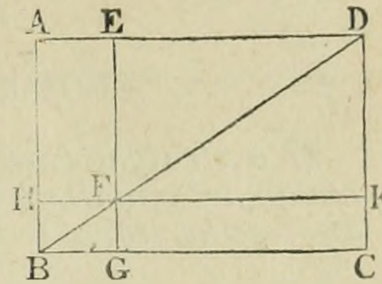
BOOK II.

DEFINITIONS.

1. Every right-angled parallelogram, or rectangle, is said to be contained by any two of the straight lines which contain one of the right angles.

2. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon.

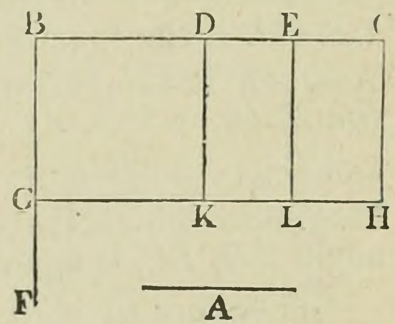
Thus the parallelogram *HG*, together with the complements *AF*, *FC*, is the gnomon, which is more briefly expressed by the letters *AGK*, or *EHC*, which are at the opposite angles of the parallelograms which make the gnomon.



PROPOSITION 1. *THEOREM.*

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let *A* and *BC* be two straight lines; and let *BC* be divided into any number of parts at the points *D*, *E*: the rectangle contained by the straight lines *A*, *BC*, shall be equal to the rectangle contained by *A*, *BD*, together with that contained by *A*, *DE*, and that contained by *A*, *EC*.



From the point *B* draw *BF* at right angles to *BC*; [I. 11.
and make *BG* equal to *A*; [I. 3.
through *G* draw *GH* parallel to *BC*; and through *D*, *E*, *C* draw *DK*, *EL*, *CH*, parallel to *BG*. [I. 31.

Then the rectangle *BH* is equal to the rectangles *BK*, *DL*, *EH*.

But *BH* is contained by *A*, *BC*, for it is contained by *GB*, *BC*, and *GB* is equal to *A*. [*Construction.*

And *BK* is contained by *A*, *BD*, for it is contained by *GB*, *BD*, and *GB* is equal to *A*;

and *DL* is contained by *A*, *DE*, because *DK* is equal to *BG*, which is equal to *A*; [I. 34.

and in like manner *EH* is contained by *A*, *EC*.

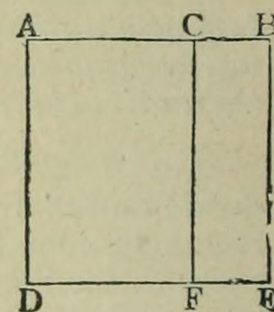
Therefore the rectangle contained by *A*, *BC* is equal to the rectangles contained by *A*, *BD*, and by *A*, *DE*, and by *A*, *EC*.

Wherefore, *if there be two straight lines* &c. Q.E.D.

PROPOSITION 2. *THEOREM.*

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.

Let the straight line AB be divided into any two parts at the point C: the rectangle contained by AB, BC, together with the rectangle AB, AC, shall be equal to the square on AB.



[*Note.* To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines AB, AC is sometimes simply called the rectangle AB, AC.]

On AB describe the square $ADEB$; [I. 46.

and through C draw CF parallel to AD or BE. [I. 31.

Then AE is equal to the rectangles AF, CE.

But AE is the square on AB.

And AF is the rectangle contained by BA, AC, for it is contained by DA, AC, of which DA is equal to BA;

and CE is contained by AB, BC, for BE is equal to AB.

Therefore the rectangle AB, AC, together with the rectangle AB, BC, is equal to the square on AB.

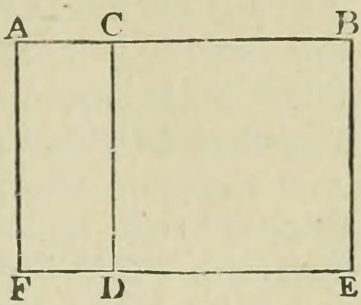
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 3. *THEOREM.*

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square on the aforesaid part.

Let the straight line AB be divided into any two parts at the point C: the rectangle AB, BC shall be equal to the rectangle AC, CB, together with the square on BC.

On BC describe the square $CDEB$; [I. 46.



produce ED to F, and through A draw AF parallel to CD or BE. [I. 31.

Then the rectangle AE is equal to the rectangles AD, CE.

But AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC;

and AD is contained by AC, CB, for CD is equal to CB; and CE is the square on BC.

Therefore the rectangle AB, BC is equal to the rectangle AC, CB, together with the square on BC.

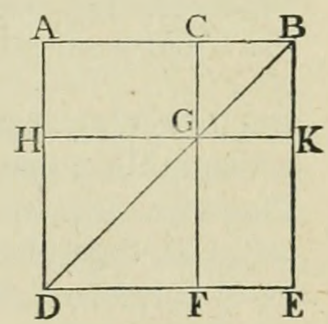
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 4. *THEOREM.*

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts at the point C: the square on AB shall be equal to the squares on AC, CB, together with twice the rectangle contained by AC, CB.

On AB describe the square $ADEB$; [I. 46.



join BD; through C draw CGF parallel to AD or BE, and through G draw HK parallel to AB or DE. [I. 31.

Then, because CF is parallel to AD, and BD falls on them, the exterior angle CGB is equal to the interior and opposite angle ADB; [I. 29.

but the angle ADB is equal to the angle ABD, [I. 5.

because BA is equal to AD, being sides of a square;

therefore the angle CGB is equal to the angle CBG; [*Ax.* 1.

and therefore the side CG is equal to the side CB. [I. 6.

But CB is also equal to GK, and CG to BK; [I. 34.

therefore the figure $CGKB$ is equilateral.

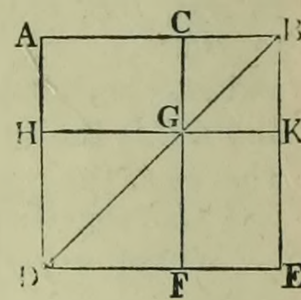
It is likewise rectangular. For since CG is parallel to BK, and CB meets them, the angles KBC, GCB are together equal to two right angles. [I. 29.

But KBC is a right angle. [I. *Definition* 30

Therefore GCB is a right angle. [*Axiom* 3.

And therefore also the angles CGK, GKB opposite to these are right angles. [I. 34. and *Axiom* 1.

Therefore $CGKB$ is rectangular; and it has been shewn to be equilateral; therefore it is a square, and it is on the side CB.



For the same reason HF is also a square, and it is on the side HG, which is equal to AC. [I. 34.

Therefore HF, CK are the squares on AC, CB.

And because the complement AG is equal to the complement GE; [I. 43.

and that AG is the rectangle contained by AC, CB, for CG is equal to CB;

therefore GE is also equal to the rectangle AC, CB. [*Ax.* 1.

Therefore AG, GE are equal to twice the rectangle AC, CB.

And HF, CK are the squares on AC, CB.

Therefore the four figures HF, CK, AG, GE are equal to the squares on AC, CB, together with twice the rectangle AC, CB.

But HF, CK, AG, GE make up the whole figure $ADEB$, which is the square on AB.

Therefore the square on AB is equal to the squares on AC, CB, together with twice the rectangle AC, CB.

Wherefore, *if a straight line* &c. Q.E.D.

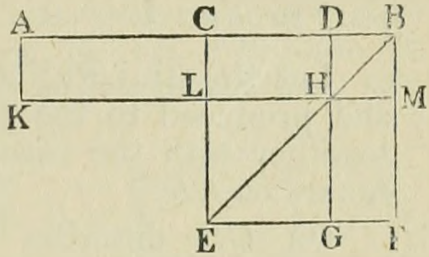
COROLLARY. From the demonstration it is manifest, that parallellograms about the diameter of a square are likewise squares.

PROPOSITION 5. *THEOREM.*

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the

unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line *AB* be divided into two equal parts at the point *C*, and into two unequal parts at the point *D*: the rectangle *AD*, *DB*, together with the square on *CD*, shall be equal to the square on *CB*.



On *CB* describe the square *CEFB*; [I. 46.
join *BE*; through *D* draw *DHG* parallel to *CE* or *BF*; through *H* draw *KLM* parallel to *CB* or *EF*; and through *A* draw *AK* parallel to *CL* or *BM*. [I. 31.

Then the complement *CH* is equal to the complement *HF*; [I. 43.
to each of these add *DM*; therefore the whole *CM* is equal to the whole *DF*. [*Axiom* 2
But *CM* is equal to *AL*, [I. 36.
because *AC* is equal to *CB*. [*Hypothesis*.
Therefore also *AL* is equal to *DF*. [*Axiom* 1.
To each of these add *CH*; therefore the whole *AH* is equal to *DF* and *CH*. [*Axiom* 2.
But *AH* is the rectangle contained by *AD*, *DB*, for *DH* is equal to *DB*; [II. 4, *Corollary*.
and *DF* together with *CH* is the gnomon *CMG*;
therefore the gnomon *CMG* is equal to the rectangle *AD*, *DB*.
To each of these add *LG*, which is equal to the square on *CD*. [II. 4, *Corollary*, and I. 34.
Therefore the gnomon *CMG*, together with *LG*, is equal to the rectangle *AD*, *DB*, together with the square on *CD*. [*Ax*. 2.
But the gnomon *CMG* and *LG* make up the whole figure *CEFB*, which is the square on *CB*.
Therefore the rectangle *AD*, *DB*, together with the square on *CD*, is equal to the square on *CB*.

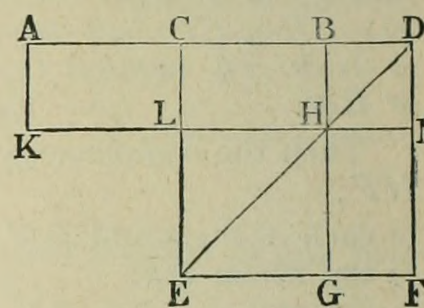
Wherefore, *if a straight line* &c. Q.E.D.

From this proposition it is manifest that the difference of the squares on two unequal straight lines *AC*, *CD*, is equal to the rectangle contained by their sum and difference.

PROPOSITION 6. *THEOREM.*

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the straight line *AB* be bisected at the point *C*, and produced to the point *D*: the rectangle *AD*, *DB*, together with the square on *CB*, shall be equal to the square on *CD*.



On *CD* describe the square *CEFD*; [I. 46.
join *DE*; through *B* draw *BHG* parallel to *CE* or *DF*; through *H* draw *KLM* parallel to *AD* or *EF*; and through *A* draw *AK* parallel to *CL* or *DM*. [I. 31.

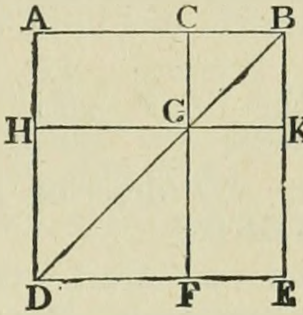
Then, because *AC* is equal to *CB*, [*Hypothesis.*
the rectangle *AL* is equal to the rectangle *CH*; [I. 36.
but *CH* is equal to *HF*; [I. 43.
therefore also *AL* is equal to *HF*. [*Axiom* 1.
To each of these add *CM*;
therefore the whole *AM* is equal to the gnomon *CMG*. [*Ax.* 2.
But *AM* is the rectangle contained by *AD*, *DB*, for *DM* is equal to *DB*. [II. 4, *Corollary.*
Therefore the rectangle *AD*, *DB* is equal to the gnomon *CMG*. [*Axiom* 1.
To each of these add *LG*, which is equal to the square on *CB*. [II. 4, *Corollary*, and I. 34.
Therefore the rectangle *AD*, *DB*, together with the square on *CB*, is equal to the gnomon *CMG* and the figure *LG*.
But the gnomon *CMG* and *LG* make up the whole figure *CEFD*, which is the square on *CD*.
Therefore the rectangle *AD*, *DB*, together with the square on *CB*, is equal to the square on *CD*.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 7. THEOREM.

If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts at the point C: the squares on AB, BC shall be equal to twice the rectangle AB, BC, together with the square on AC.



On AB describe the square $ADEB$, and construct the figure as in the preceding propositions.

Then AG is equal to GE; [I. 43.

to each of these add CK;

therefore the whole AK is equal to the whole CE;

therefore AK, CE are double of AK.

But AK, CE are the gnomon AKF, together with the square CK;

therefore the gnomon AKF, together with the square CK, is double of AK.

But twice the rectangle AB, BC is double of AK, for BK is equal to BC. [II. 4, *Corollary.*

Therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB, BC.

To each of these equals add HF, which is equal to the square on AC. [II. 4, *Corollary*, and I. 34.

Therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB, BC, together with the square on AC.

But the gnomon AKF together with the squares CK, HF, make up the whole figure $ADEB$ and CK, which are the squares on AB and BC.

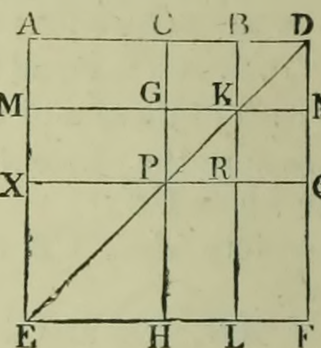
Therefore the squares on AB, BC, are equal to twice the rectangle AB, BC, together with the square on AC.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 8. *THEOREM.*

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

Let the straight line AB be divided into any two parts at the point C: four times the rectangle AB, BC, together with the square on AC, shall be equal to the square on the straight line made up of AB and BC together.



Produce AB to D, so that BD may be equal to CB; [*Post.* 2. and I. 3.

on AD describe the square $AEFD$;

and construct two figures such as in the preceding propositions.

Then, because CB is equal to BD, [*Construction.*

and that CB is equal to GK, and BD to KN, [I. 34.

therefore GK is equal to KN. [*Axiom* 1.

For the same reason PR is equal to RO.

And because CB is equal to BD, and GK to KN, the rectangle CK is equal to the rectangle BN, and the rectangle GR to the rectangle RN. [I. 36.

But CK is equal to RN, because they are the complements of the parallelogram CO; [I. 43.

therefore also BN is equal to GR. [*Axiom* 1.

Therefore the four rectangles BN, CK, GR, RN are equal to one another, and so the four are quadruple of one of them CK.

Again, because CB is equal to BD, [*Construction.*

and that BD is equal to BK, [II. 4, *Corollary.*

that is to CG, [I. 34.

and that CB is equal to GK, [I. 34.

that is to GP: [II. 4, *Corollary.*

therefore CG is equal to GP. [*Axiom* 1.

And because CG is equal to GP, and PR to RO, the rectangle AG is equal to the rectangle MP, and the rectangle PL to the rectangle RF. [I. 36.

But MP is equal to PL, because they are the complements of the parallelogram ML; [I. 43.

therefore also AG is equal to RF. [*Axiom* 1.

Therefore the four rectangles AG, MP, PL, RF are equal to one another, and so the four are quadruple of one of them AG.

And it was shewn that the four CK, BN, GR and RN are quadruple of CK; therefore the eight rectangles which make up the gnomon AOH are quadruple of AK.

And because AK is the rectangle contained by AB, BC, for BK is equal to BC;

therefore four times the rectangle AB, BC is quadruple of AK.

But the gnomon AOH was shewn to be quadruple of AK.

Therefore four times the rectangle AB, BC is equal to the gnomon AOH. [*Axiom* 1.

To each of these add XH, which is equal to the square on AC. [II. 4, *Corollary*, and I. 34.

Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the gnomon AOH and the square XH.

But the gnomon AOH and the square XH make up the figure $AEFD$, which is the square on AD.

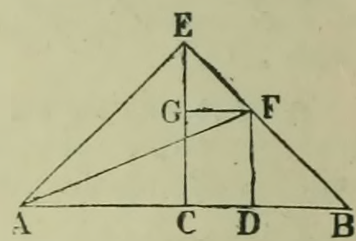
Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the square on AD, that is to the square on the line made of AB and BC together.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 9. *THEOREM.*

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D: the squares on AD, DB shall be together double of the squares on AC, CD.



From the point C draw CE at right angles to AB, [I. 11. and make it equal to AC or CB, [I. 3. and join EA, EB; through D draw DF parallel to CE, and through F draw FG parallel to BA; [I. 31. and join AF.

Then, because AC is equal to CE, [*Construction.*

the angle EAC is equal to the angle AEC. [I. 5.

And because the angle ACE is a right angle, [*Construction.*

the two other angles AEC, EAC are together equal to one right angle; [I. 32.

and they are equal to one another;

therefore each of them is half a right angle.

For the same reason each of the angles CEB, EBC is half a right angle.

Therefore the whole angle AEB is a right angle.

And because the angle GEF is half a right angle, and the angle EGF a right angle, for it is equal to the interior and opposite angle ECB; [I. 29.

therefore the remaining angle EFG is half a right angle.

Therefore the angle GEF is equal to the angle EFG, and the side EG is equal to the side GF. [I. 6.

Again, because the angle at B is half a right angle, and the

angle FDB a right angle, for it is equal to the interior and opposite angle ECB; [I. 29.
therefore the remaining angle BFD is half a right angle. Therefore the angle at B is equal to the angle BFD, and the side DF is equal to the side DB. [I. 6.

And because AC is equal to CE, [*Construction.*
the square on AC is equal to the square on CE;
therefore the squares on AC, CE are double of the square on AC.
But the square on AE is equal to the squares on AC, CE, because the angle ACE is a right angle; [I. 47.
therefore the square on AE is double of the square on AC.
Again, because EG is equal to GF, [*Construction.*
the square on EG is equal to the square on GF;
therefore the squares on EG, GF are double of the square on GF.
But the square on EF is equal to the squares on EG, GF, because the angle EGF is a right angle; [I. 47.
therefore the square on EF is double of the square on GF.
And GF is equal to CD; [I. 34.
therefore the square on EF is double of the square on CD.
But it has been shewn that the square on AE is also double of the square on AC.
Therefore the squares on AE, EF are double of the squares on AC, CD.

But the square on AF is equal to the squares on AE, EF, because the angle AEF is a right angle. [I. 47.
Therefore the square on AF is double of the squares on AC, CD.
But the squares on AD, DF are equal to the square on AF, because the angle ADF is a right angle. [I. 47.
Therefore the squares on AD, DF are double of the squares on AC, CD.
And DF is equal to DB;
therefore the squares on AD, DB are double of the squares on AC, CD.

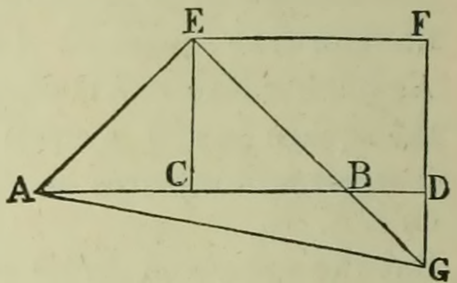
Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 10. *THEOREM.*

If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.

Let the straight line AB be bisected at C, and produced to D: the squares on AD, DB shall be together double of the squares on AC, CD.

From the point C draw CE at right angles to AB, [I. 11. and make it equal to AC or CB; [I. 3. and join AE, EB; through E draw EF parallel to AB, and through D draw DF parallel to CE. [I. 31.



Then because the straight line EF meets the parallels EC, FD, the angles CEF, EFD are together equal to two right angles; [I. 29.

and therefore the angles BEF, EFD are together less than two right angles.

Therefore the straight lines EB, FD will meet, if produced, towards B, D. [*Axiom* 12.

Let them meet at G, and join AG.

Then because AC is equal to CE, [*Construction.*
the angle CEA is equal to the angle EAC; [I. 5.
and the angle ACE is a right angle; [*Construction.*
therefore each of the angles CEA, EAC is half a right angle. [I. 32.

For the same reason each of the angles CEB, EBC is half a right angle.

Therefore the angle AEB is a right angle.

And because the angle EBC is half a right angle, the angle DBG is also half a right angle, for they are vertically opposite; [I. 15.
but the angle BDG is a right angle, because it is equal to the alternate angle DCE; [I. 29.
therefore the remaining angle DGB is half a right angle, [I. 32.

and is therefore equal to the angle DBG;
therefore also the side BD is equal to the side DG. [I. 6.
Again, because the angle EGF is half a right angle, and the angle at F a right angle, for it is equal to the opposite angle ECD; [I. 34.
therefore the remaining angle FEG is half a right angle, [I. 32.
and is therefore equal to the angle EGF;
therefore also the side GF is equal to the side FE. [I. 6.

And because EC is equal to CA, the square on EC is equal to the square on CA;
therefore the squares on EC, CA are double of the square on CA.
But the square on AE is equal to the squares on EC, CA. [I. 47.
Therefore the square on AE is double of the square on AC.
Again, because GF is equal to FE, the square on GF is equal to the square on FE;
therefore the squares on GF, FE are double of the square on FE.
But the square on EG is equal to the squares on GF, FE. [I. 47.
Therefore the square on EG is double of the square on FE.
And FE is equal to CD; [I. 34.
therefore the square on EG is double of the square on CD.
But it has been shewn that the square on AE is double of the square on AC.
Therefore the squares on AE, EG are double of the squares on AC, CD.

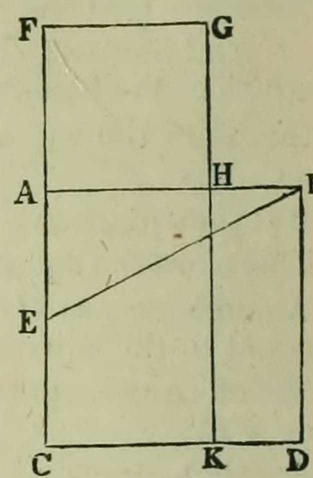
But the square on AG is equal to the squares on AE, EG. [I. 47.
Therefore the square on AG is double of the squares on AC, CD.
But the squares on AD, DG are equal to the square on AG. [I. 47.
Therefore the squares on AD, DG are double of the squares on AC, CD.
And DG is equal to DB;
therefore the squares on AD, DB are double of the squares on AC, CD.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 11. *PROBLEM.*

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

Let AB be the given straight line: it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.



On AB describe the square $ABDC$; [I. 46.

bisect AC at E; [I. 10.

join BE; produce CA to F, and make EF equal to EB; [I. 3.

and on AF describe the square $AFGH$. [I. 46.

AB shall be divided at H so that the rectangle AB, BH is equal to the square on AH.

Produce GH to K.

Then, because the straight line AC is bisected at E, and produced to F, the rectangle CF, FA, together with the square on AE, is equal to the square on EF. [II. 6.

But EF is equal to EB. [*Construction.*

Therefore the rectangle CF, FA, together with the square on AE, is equal to the square on EB.

But the square on EB is equal to the squares on AE, AB, because the angle EAB is a right angle. [I. 47.

Therefore the rectangle CF, FA, together with the square on AE, is equal to the squares on AE, AB.

Take away the square on AE, which is common to both; therefore the remainder, the rectangle CF, FA, is equal to the square on AB. [*Axiom* 3.

But the figure FK is the rectangle contained by CF, FA, for FG is equal to FA;

and AD is the square on AB;

therefore FK is equal to AD.

Take away the common part AK, and the remainder FH is equal to the remainder HD. [*Axiom* 3.

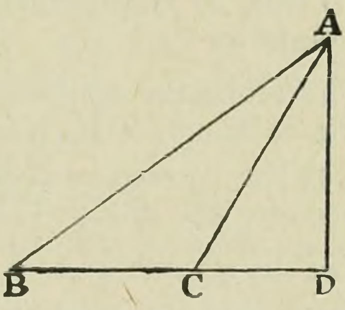
But HD is the rectangle contained by AB, BH, for AB is equal to BD;
and FH is the square on AH;
therefore the rectangle AB, BH is equal to the square on AH.

Wherefore *the straight line AB is divided at H, so that the rectangle AB, BH is equal to the square on AH.* Q.E.F.

PROPOSITION 12. *THEOREM.*

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn perpendicular to BC produced: the square on AB shall be greater than the squares on AC, CB, by twice the rectangle BC, CD.



Because the straight line BD is divided into two parts at the point C, the square on BD is equal to the squares on BC, CD, and twice the rectangle BC, CD. [II. 4.

To each of these equals add the square on DA.

Therefore the squares on BD, DA are equal to the squares on BC, CD, DA, and twice the rectangle BC, CD. [*Axiom* 2.

But the square on BA is equal to the squares on BD, DA, because the angle at D is a right angle; [I. 47.

and the square on CA is equal to the squares on CD, DA. [I. 47.

Therefore the square on BA is equal to the squares on BC, CA, and twice the rectangle BC, CD;

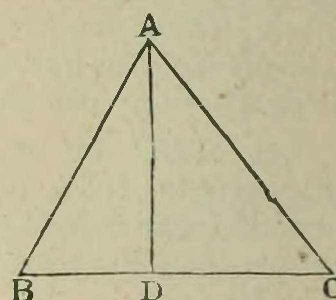
that is, the square on BA is greater than the squares on BC, CA by twice the rectangle BC, CD.

Wherefore, *in obtuse-angled triangles* &c. Q.E.D.

PROPOSITION 13. *THEOREM.*

In every triangle, the square on the side subtending an acute angle, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.

Let *ABC* be any triangle, and the angle at *B* an acute angle; and on *BC* one of the sides containing it, let fall the perpendicular *AD* from the opposite angle: the square on *AC*, opposite to the angle *B*, shall be less than the squares on *CB*, *BA*, by twice the rectangle *CB*, *BD*.



First, let *AD* fall within the triangle *ABC*.

Then, because the straight line *CB* is divided into two parts at the point *D*, the squares on *CB*, *BD* are equal to twice the rectangle contained by *CB*, *BD* and the square on *CD*. [II. 7.

To each of these equals add the square on *DA*.

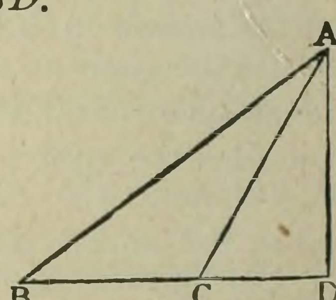
Therefore the squares on *CB*, *BD*, *DA* are equal to twice the rectangle *CB*, *BD* and the squares on *CD*, *DA*. [*Ax.* 2.

But the square on *AB* is equal to the squares on *BD*, *DA*, because the angle *BDA* is a right angle; [I. 47.

and the square on *AC* is equal to the squares on *CD*, *DA*. [I. 47.

Therefore the squares on *CB*, *BA* are equal to the square on *AC* and twice the rectangle *CB*, *BD*;

that is, the square on *AC* alone is less than the squares on *CB*, *BA* by twice the rectangle *CB*, *BD*.



Secondly, let *AD* fall without the triangle *ABC*.

Then because the angle at *D* is a right angle, [*Construction.*

the angle *ACB* is greater than a right angle; [I. 16.

and therefore the square on AB is equal to the squares on AC, CB, and twice the rectangle BC, CD. [II. 12.

To each of these equals add the square on BC.

Therefore the squares on AB, BC are equal to the square on AC, and twice the square on BC, and twice the rectangle BC, CD. [*Axiom* 2.

But because BD is divided into two parts at C, the rectangle DB, BC is equal to the rectangle BC, CD and the square on BC; [II. 3.

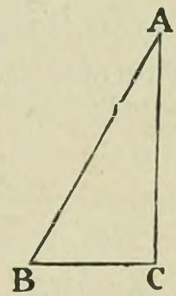
and the doubles of these are equal,

that is, twice the rectangle DB, BC is equal to twice the rectangle BC, CD and twice the square on BC.

Therefore the squares on AB, BC are equal to the square on AC, and twice the rectangle DB, BC;

that is, the square on AC alone is less than the squares on AB, BC by twice the rectangle DB, BC.

Lastly, let the side AC be perpendicular to BC.



Then BC is the straight line between the perpendicular and the acute angle at B;

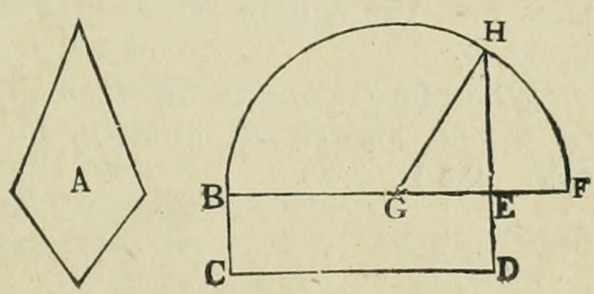
and it is manifest, that the squares on AB, BC are equal to the square on AC, and twice the square on BC. [I. 47 and *Ax.* 2.

Wherefore, *in every triangle* &c. Q.E.D.

PROPOSITION 14. *PROBLEM.*

To describe a square that shall be equal to a given rectilineal figure.

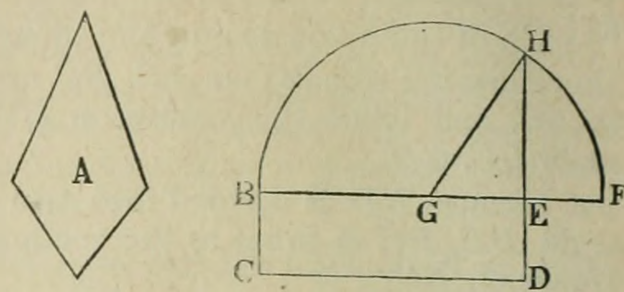
Let A be the given rectilineal figure: it is required to describe a square that shall be equal to A.



Describe the rectangular parallelogram $BCDE$ equal to the rectilineal figure A. [I. 45.

Then if the sides of it, BE, ED are equal to one another, it is a square, and what was required is now done.

But if they are not equal, produce one of them BE to F, make EF equal to ED, [I. 3.
and bisect BF at G; [I. 10.
from the centre G, at the distance GB, or GF, describe the semicircle BHF, and produce DE to H.



The square described on EH shall be equal to the given rectilineal figure A.

Join GH. Then, because the straight line BF is divided into two equal parts at the point G, and into two unequal parts at the point E, the rectangle BE, EF, together with the square on GE, is equal to the square on GF. [II. 5.
But GF is equal to GH.
Therefore the rectangle BE, EF, together with the square on GE, is equal to the square on GH.
But the square on GH is equal to the squares on GE, EH; [I. 47.
therefore the rectangle BE, EF, together with the square on GE, is equal to the squares on GE, EH.
Take away the square on GE, which is common to both;
therefore the rectangle BE, EF is equal to the square on EH. [*Axiom* 3.
But the rectangle contained by BE, EF is the parallelogram BD,
because EF is equal to ED. [*Construction.*
Therefore BD is equal to the square on EH.
But BD is equal to the rectilineal figure A. [*Construction.*
Therefore the square on EH is equal to the rectilineal figure A.

Wherefore *a square has been made equal to the given rectilineal figure* A, *namely, the square described on* EH. Q.E.F.

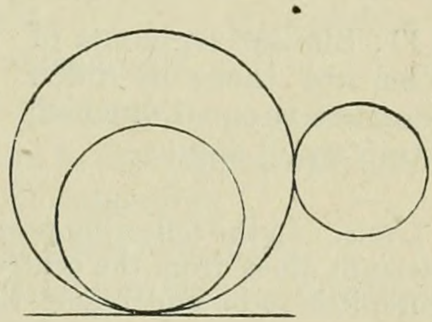
BOOK III.

DEFINITIONS.

1. Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

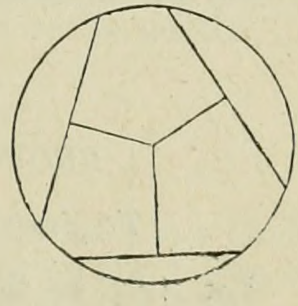
This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.

2. A straight line is said to touch a circle, when it meets the circle, and being produced does not cut it.



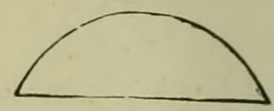
3. Circles are said to touch one another, which meet but do not cut one another.

4. Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



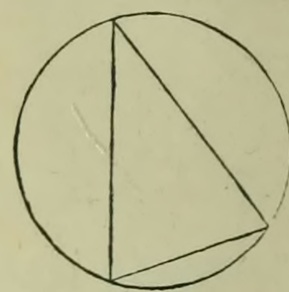
5. And the straight line on which the greater perpendicular falls, is said to be farther from the centre.

6. A segment of a circle is the figure contained by a straight line and the circumference it cuts off.



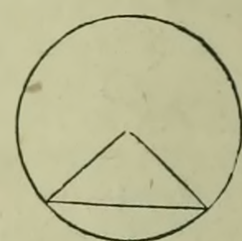
7. The angle of a segment is that which is contained by the straight line and the circumference.

8. An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.

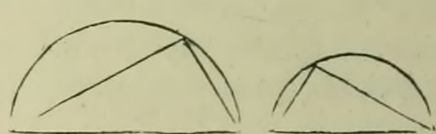


9. And an angle is said to insist or stand on the circumference intercepted between the straight lines which contain the angle.

10. A sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



11. Similar segments of circles are those in which the angles are equal, or which contain equal angles.



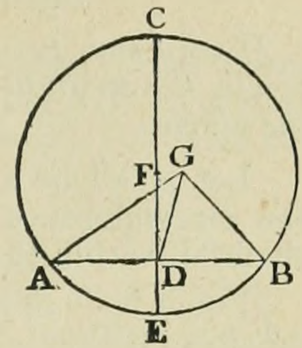
[*Note.* In the following propositions, whenever the expression "straight lines from the centre," or "drawn from the centre," occurs, it is to be understood that the lines are drawn to the circumference.

Any portion of the circumference is called an *arc.*]

PROPOSITION 1. *PROBLEM.*

To find the centre of a given circle.

Let ABC be the given circle: it is required to find its centre.



Draw within it any straight line AB, and bisect AB at D; [I. 10.
from the point D draw DC at right angles to AB; [I. 11.
produce CD to meet the circumference at E, and bisect CE at F. [I. 10.
The point F shall be the centre of the circle ABC.

For if F be not the centre, if possible, let G be the centre; and join GA, GD, GB.
Then, because DA is equal to DB, [*Construction.*
and DG is common to the two triangles ADG, BDG;
the two sides AD, DG are equal to the two sides BD, DG, each to each;
and the base GA is equal to the base GB, because they are drawn from the centre G; [I. *Definition* 15.
therefore the angle ADG is equal to the angle BDG. [I. 8.
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [I. *Definition* 10.
therefore the angle BDG is a right angle.
But the angle BDF is also a right angle. [*Construction.*
Therefore the angle BDG is equal to the angle BDF, [*Ax.* 11.
the less to the greater; which is impossible.
Therefore G is not the centre of the circle ABC.

In the same manner it may be shewn that no other point out of the line CE is the centre;
and since CE is bisected at F, any other point in CE divides it into unequal parts, and cannot be the centre.
Therefore no point but F is the centre;
that is, F is the centre of the circle ABC:
which was to be found.

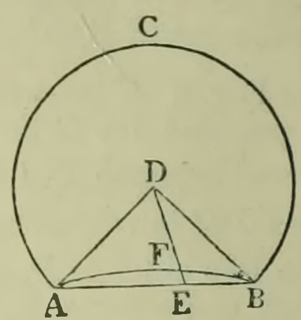
Corollary. From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the straight line which bisects the other.

PROPOSITION 2. *THEOREM.*

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A and B any two points in the circumference: the straight line drawn from A to B shall fall within the circle.

For if it do not, let it fall, if possible, without, as AEB. Find D the centre of the circle ABC; [III. 1.
and join DA, DB; in the arc AB take any point F, join DF, and produce it to meet the straight line AB at E.



Then, because DA is equal to DB, [I. *Definition* 15.
the angle DAB is equal to the angle DBA. [I. 5.
And because AE, a side of the triangle DAE, is produced to B, the exterior angle DEB is greater than the interior opposite angle DAE. [I. 16.
But the angle DAE was shewn to be equal to the angle DBE;
therefore the angle DEB is greater than the angle DBE.
But the greater angle is subtended by the greater side; [I. 19.
therefore DB is greater than DE.
But DB is equal to DF; [I. *Definition* 15.
therefore DF is greater than DE, the less than the greater; which is impossible.
Therefore the straight line drawn from A to B does not fall without the circle.

In the same manner it may be shewn that it does not fall on the circumference.
Therefore it falls within the circle.

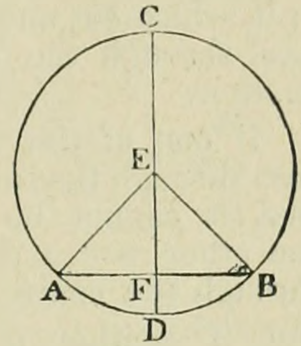
Wherefore, *if any two points* &c. Q.E.D.

PROPOSITION 3. *THEOREM.*

*If a straight line drawn through the centre of a circle, bisect a straight line in it which does not pass through the

centre, it shall cut it at right angles; and if it cut it at right angles it shall bisect it.

Let ABC be a circle; and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, at the point F: CD shall cut AB at right angles.



Take E the centre of the circle; and join EA, EB. [III. 1.

Then, because AF is equal to FB, [*Hypothesis.*

and FE is common to the two triangles AFE, BFE;

the two sides AF, FE are equal to the two sides BF, FE, each to each;

and the base EA is equal to the base EB; [I. *Def.* 15.

therefore the angle AFE is equal to the angle BFE. [I. 8.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [I. *Definition* 10.

therefore each of the angles AFE, BFE is a right angle.

Therefore the straight line CD, drawn through the centre, bisecting another AB which does not pass through the centre, also cuts it at right angles.

But let CD cut AB at right angles: CD shall also bisect AB; that is, AF shall be equal to FB.

The same construction being made, because EA, EB, drawn from the centre, are equal to one another, [I. *Def.* 15.

the angle EAF is equal to the angle EBF. [I. 5.

And the right angle AFE is equal to the right angle BFE.

Therefore in the two triangles EAF, EBF, there are two angles in the one equal to two angles in the other, each to each;

and the side EF, which is opposite to one of the equal angles in each, is common to both;

therefore their other sides are equal; [I. 26.

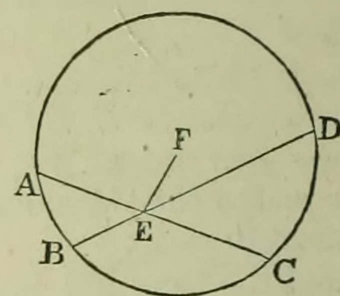
therefore AF is equal to FB.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 4. *THEOREM.*

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect one another.

Let *ABCD* be a circle, and *AC*, *BD* two straight lines in it, which cut one another at the point *E*, and do not both pass through the centre: *AC*, *BD* shall not bisect one another.



If one of the straight lines pass through the centre it is plain that it cannot be bisected by the other which does not pass through the centre.

But if neither of them pass through the centre, if possible, let *AE* be equal to *EC*, and *BE* equal to *ED*.

Take *F* the centre of the circle [III. 1.
and join *EF*.

Then, because *FE*, a straight line drawn through the centre, bisects another straight line *AC* which does not pass through the centre; [*Hypothesis.*

FE cuts *AC* at right angles; [III. 3.

therefore the angle *FEA* is a right angle.

Again, because the straight line *FE* bisects the straight line *BD*, which does not pass through the centre, [*Hyp.*

FE cuts *BD* at right angles; [III. 3.

therefore the angle *FEB* is a right angle.

But the angle *FEA* was shewn to be a right angle;

therefore the angle *FEA* is equal to the angle *FEB*, [*Ax.* 11.

the less to the greater; which is impossible.

Therefore *AC*, *BD* do not bisect each other.

Wherefore, *if in a circle* &c. Q.E.D.

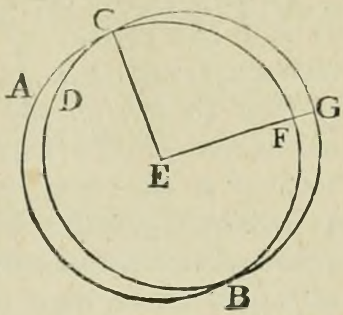
PROPOSITION 5. *THEOREM.*

If two circles cut one another, they shall not have the same centre.

Let the two circles *ABC*, *CDG* cut one another at the

points B, C: they shall not have the same centre.

For, if it be possible, let E be their centre; join EC, and draw any straight line EFG meeting the circumferences at F and G.



Then, because E is the centre of the circle ABC, EC is equal to EF. [I. *Definition* 15.

Again, because E is the centre of the circle CDG, EC is equal to EG. [I. *Definition* 15.

But EC was shewn to be equal to EF;

therefore EF is equal to EG, [*Axiom* 1.

the less to the greater; which is impossible.

Therefore E is not the centre of the circles ABC, CDG.

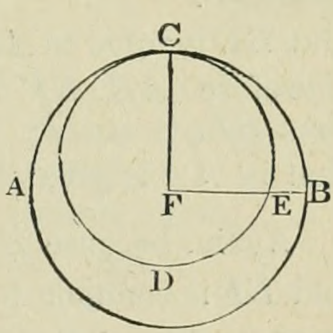
Wherefore, *if two circles* &c. Q.E.D.

PROPOSITION 6. *THEOREM.*

If two circles touch one another internally, they shall not have the same centre.

Let the two circles ABC, CDE touch one another internally at the point C: they shall not have the same centre.

For, if it be possible, let F be their centre; join FC, and draw any straight line FEB, meeting the circumferences at E and B.



Then, because F is the centre of the circle ABC, FC is equal to FB. [I. *Def.* 15.

Again, because F is the centre of the circle CDE, FC is equal to FE. [I. *Definition* 15.

But FC was shewn to be equal to FB;

therefore FE is equal to FB, [*Axiom* 1.

the less to the greater; which is impossible.

Therefore F is not the centre of the circles ABC, CDE.

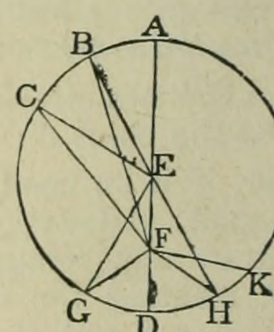
Wherefore, *if two circles* &c. Q.E.D

PROPOSITION 7. *THEOREM.*

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from this point to the circumference, the greatest is that in which the centre is, and the other part of the diameter is the least; and, of any others, that which is nearer to the straight line which passes through the centre, is always greater than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.

Let $ABCD$ be a circle and AD its diameter, in which let any point F be taken which is not the centre; let E be the centre: of all the straight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA, which passes through E, shall be the greatest, and FD, the other part of the diameter AD, shall be the least; and of the others FB shall be greater than FC, and FC than FG.

Join BE, CE, GE.



Then, because any two sides of a triangle are greater than the third side, [I. 20.

therefore BE, EF are greater than BF.

But BE is equal to AE; [I. *Def.* 15.

therefore AE, EF are greater than BF,

that is, AF is greater than BF.

Again, because BE is equal to CE, [I. *Definition* 15.

and EF is common to the two triangles BEF, CEF;

the two sides BE, EF are equal to the two sides CE, EF, each to each;

but the angle BEF is greater than the angle CEF;

therefore the base FB is greater than the base FC. [I. 24.

In the same manner it may be shewn that FC is greater than FG.

Again, because GF, FE are greater than EG, [I. 20.

and that EG is equal to ED; [I. *Definition* 15.
therefore GF, FE are greater than ED.
Take away the common part FE, and the remainder GF is greater than the remainder FD.

Therefore FA is the greatest, and FD the least of all the straight lines from F to the circumference; and FB is greater than FC, and FC than FG.

Also, there can be drawn two equal straight lines from the point F to the circumference, one on each side of the shortest line FD.

For, at the point E, in the straight line EF, make the angle FEH equal to the angle FEG, [I. 23.
and join FH.

Then, because EG is equal to EH, [I. *Definition* 15.
and EF is common to the two triangles GEF, HEF;
the two sides EG, EF are equal to the two sides EH, EF, each to each;
and the angle GEF is equal to the angle HEF; [*Constr.*
therefore the base FG is equal to the base FH. [I. 4.

But, besides FH, no other straight line can be drawn from F to the circumference, equal to FG.

For, if it be possible, let FK be equal to FG.
Then, because FK is equal to FG, [*Hypothesis.*
and FH is also equal to FG,
therefore FH is equal to FK; [*Axiom* 1.
that is, a line nearer to that which passes through the centre is equal to a line which is more remote;
which is impossible by what has been already shewn.

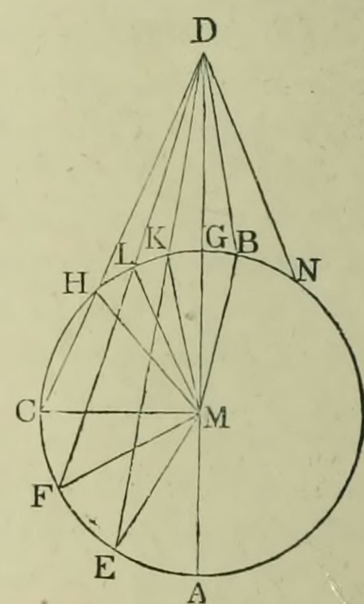
Wherefore, *if any point be taken* &c. Q.E.D.

PROPOSITION 8. *THEOREM.*

If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre; of those which fall on the concave circumference, the greatest is that which passes through the centre, and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote; but of those which fall on the

convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.

Let ABC be a circle, and D any point without it, and from D let the straight lines DA, DE, DF, DC be drawn to the circumference, of which DA passes through the centre: of those which fall on the concave circumference $AEFC$, the greatest shall be DA which passes through the centre, and the nearer to it shall be greater than the more remote, namely, DE greater than DF, and DF greater than DC; but of those which fall on the convex circumference $GKLH$, the least shall be DG between the point D and the diameter AG, and the nearer to it shall be less than the more remote, namely, DK less than DL, and DL less than DH.



Take M, the centre of the circle ABC, [III. 1.
and join ME, MF, MC, MH, ML, MK.

Then, because any two sides of a triangle are greater than the third side, [I. 20.
therefore EM, MD are greater than ED.
But EM is equal to AM; [I. *Def.* 15.
therefore AM, MD are greater than ED,
that is, AD is greater than ED.

Again, because EM is equal to FM,
and MD is common to the two triangles EMD, FMD;
the two sides EM, MD are equal to the two sides FM, MD, each to each;
but the angle EMD is greater than the angle FMD;
therefore the base ED is greater than the base FD. [I. 24.

In the same manner it may be shewn that FD is greater than CD.

Therefore DA is the greatest, and DE greater than DF, and DF greater than DC.

Again, because MK, KD are greater than MD, [I. 20.
and MK is equal to MG, [I. *Definition* 15.
the remainder KD is greater than the remainder GD,
that is, GD is less than KD.

And because MLD is a triangle, and from the points M, D, the extremities of its side MD, the straight lines MK, DK are drawn to the point K within the triangle, therefore MK, KD are less than ML, LD; [I. 21.
and MK is equal to ML; [I. *Definition* 15.
therefore the remainder KD is less than the remainder LD.

In the same manner it may be shewn that LD is less than HD.

Therefore DG is the least, and DK less than DL, and DL less than DH.

Also, there can be drawn two equal straight lines from the point D to the circumference, one on each side of the least line.

For, at the point M, in the straight line MD, make the angle DMB equal to the angle DMK, [I. 23.
and join DB

Then, because MK is equal to MB,
and MD is common to the two triangles KMD, BMD;
the two sides KM, MD are equal to the two sides BM, MD, each to each;
and the angle DMK is equal to the angle DMB; [*Constr.*
therefore the base DK is equal to the base DB. [I. 4.

But, besides DB, no other straight line can be drawn from D to the circumference, equal to DK.

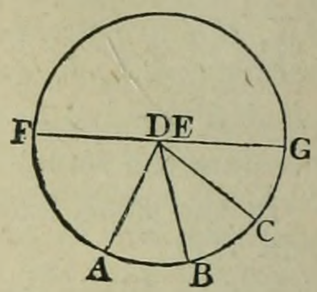
For, if it be possible, let DN be equal to DK.
Then, because DN is equal to DK,
and DB is also equal to DK,
therefore DB is equal to DN; [*Axiom* 1.
that is, a line nearer to the least is equal to one which is more remote;
which is impossible by what has been already shewn.

Wherefore, *if any point be taken* &c. Q.E.D.

PROPOSITION 9. *THEOREM.*

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal straight lines, namely DA, DB, DC: the point D shall be the centre of the circle.



For, if not, let E be the centre; join DE and produce it both ways to meet the circumference at F and G; then FG is a diameter of the circle.

Then, because in FG, a diameter of the circle ABC, the point D is taken, which is not the centre, DG is the greatest straight line from D to the circumference, and DC is greater than DB, and DB greater than DA; [III. 7.

but they are likewise equal, by hypothesis;

which is impossible.

Therefore E is not the centre of the circle ABC.

In the same manner it may be shewn that any other point than D is not the centre;

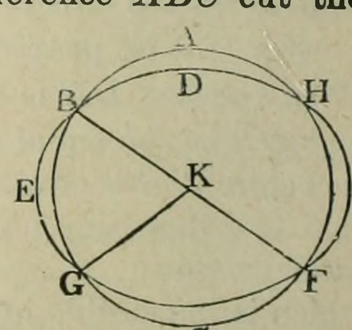
therefore D is the centre of the circle ABC.

Wherefore, *if a point be taken* &c. Q.E.D.

PROPOSITION 10. *THEOREM.*

One circumference of a circle cannot cut another at more than two points.

If it be possible, let the circumference ABC cut the circumference DEF at more than two points, namely, at the points B, G, F.



Take K, the centre of the circle ABC, [III. 1.

and join KB, KG, KF.

Then, because K is the centre of the circle ABC,

therefore KB, KG, KF are all equal to each other. [I. *Def.* 15.

And because within the circle DEF, the point K is taken, from which to the circumference DEF fall more than two equal straight lines KB, KG, KF, therefore K is the centre of the circle DEF. [III. 9.

But K is also the centre of the circle ABC. [*Construction.*

Therefore the same point is the centre of two circles which cut one another;

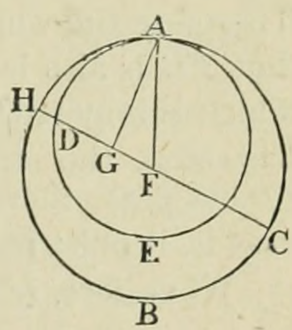
which is impossible. [III. 5.

Wherefore, *one circumference* &c. Q.E.D.

PROPOSITION 11. *THEOREM.*

If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.

Let the two circles ABC, ADE touch one another internally at the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE: the straight line which joins the centres F, G, being produced, shall pass through the point A.



For, if not, let it pass otherwise, if possible, as $FGDH$, and join AF, AG.

Then, because AG, GF are greater than AF, [I. 20.

and AF is equal to HF, [I. *Def.* 15.

therefore AG, GF, are greater than HF.

Take away the common part GF;

therefore the remainder AG is greater than the remainder HG.

But AG is equal to DG. [I. *Definition* 15.

Therefore DG is greater than HG, the less than the greater; which is impossible.

Therefore the straight line which joins the points F, G, being produced, cannot pass otherwise than through the point A,

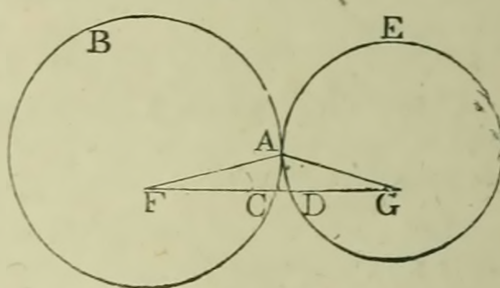
that is, it must pass through A.

Wherefore, *if two circles* &c. Q.E.D.

PROPOSITION 12. *THEOREM.*

If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.

Let the two circles *ABC*, *ADE* touch one another externally at the point *A*; and let *F* be the centre of the circle *ABC*, and *G* the centre of the circle *ADE*: the straight line which joins the points *F*, *G*, shall pass through the point *A*.



For, if not, let it pass otherwise, if possible, as *FCDG*, and join *FA*, *AG*.

Then, because *F* is the centre of the circle *ABC*, *FA* is equal to *FC*; [I. *Def.* 15.
and because *G* is the centre of the circle *ADE*, *GA* is equal to *GD*;
therefore *FA*, *AG* are equal to *FC*, *DG*. [*Axiom* 2.
Therefore the whole *FG* is greater than *FA*, *AG*.
But *FG* is also less than *FA*, *AG*; [I. 20.
which is impossible.
Therefore the straight line which joins the points *F*, *G*, cannot pass otherwise than through the point *A*,
that is, it must pass through *A*.

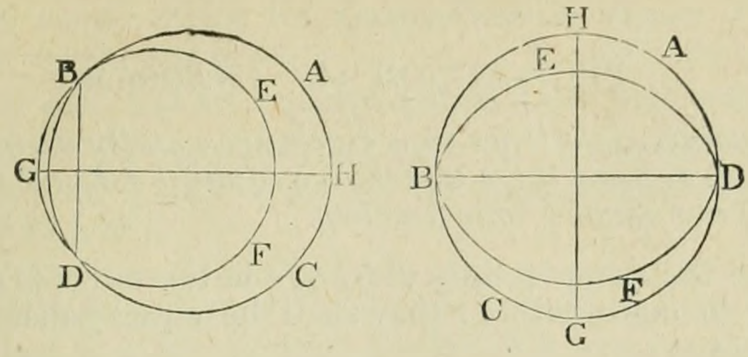
Wherefore, *if two circles* &c. Q.E.D.

PROPOSITION 13. *THEOREM.*

One circle cannot touch another at more points than one, whether it touches it on the inside or outside.

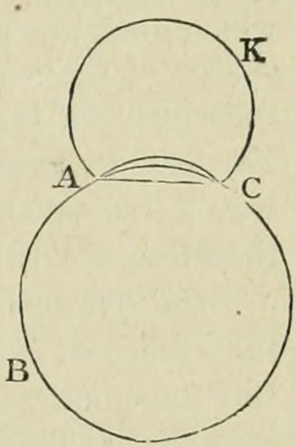
For, if it be possible, let the circle *EBF* touch the circle *ABC* at more points than one; and first on the inside, at the points *B*, *D*. Join *BD*, and draw *GH* bisecting *BD* at right angles. [I. 10, 11.

Then, because the two points *B*, *D* are in the circumference of each of the circles, the straight line *BD* falls within each of them; [III. 2.



and therefore the centre of each circle is in the straight line *GH* which bisects *BD* at right angles; [III. 1, *Corol.*
therefore *GH* passes through the point of contact. [III. 11.
But *GH* does not pass through the point of contact, because the points *B*, *D* are out of the line *GH*;
which is absurd.
Therefore one circle cannot touch another on the inside at more points than one.

Nor can one circle touch another on the outside at more points than one.



For, if it be possible, let the circle *ACK* touch the circle *ABC* at the points *A*, *C*. Join *AC*.

Then, because the two points *A*, *C* are in the circumference of the circle *ACK*, the straight line *AC* which joins them, falls within the circle *ACK*; [III. 2.
but the circle *ACK* is without the circle *ABC*; [*Hypothesis.*
therefore the straight line *AC* is without the circle *ABC*.
But because the two points *A*, *C* are in the circumference of the circle *ABC*, the straight line *AC* falls within the circle *ABC*; [III. 2.
which is absurd.
Therefore one circle cannot touch another on the outside at more points than one.

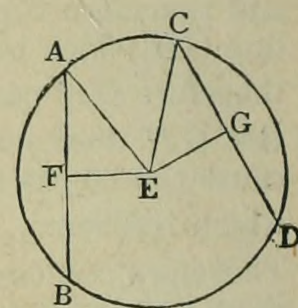
And it has been shewn that one circle cannot touch another on the inside at more points than one.

Wherefore, *one circle* &c. Q.E.D.

PROPOSITION 14. *THEOREM.*

Equal straight lines in a circle are equally distant from the centre: and those which are equally distant from the centre are equal to one another.

Let the straight lines AB, CD in the circle $ABDC$, be equal to one another: they shall be equally distant from the centre.



Take E, the centre of the circle $ABDC$; [III. 1.

and from E draw EF, EG perpendiculars to AB, CD; [I. 12.

and join EA, EC.

Then, because the straight line EF, passing through the centre, cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects it; [III. 3.

therefore AF is equal to FB, and AB is double of AF.

For the like reason CD is double of CG.

But AB is equal to CD; [*Hypothesis.*

therefore AF is equal to CG. [*Axiom* 7.

And because AE is equal to CE, [I. *Definition* 15.

the square on AE is equal to the square on CE.

But the squares on AF, FE are equal to the square on AE, because the angle AFE is a right angle; [I. 47.

and for the like reason the squares on CG, GE are equal to the square on CE;

therefore the squares on AF, FE are equal to the squares on CG, GE. [*Axiom* 1.

But the square on AF is equal to the square on CG, because AF is equal to CG;

therefore the remaining square on FE is equal to the remaining square on GE; [*Axiom* 3.

and therefore the straight line EF is equal to the straight line EG.

But straight lines in a circle are said to be equally distant

from the centre, when the perpendiculars drawn to them from the centre are equal; [III. *Definition* 4.

therefore AB, CD are equally distant from the centre.

Next, let the straight lines AB, CD be equally distant from the centre, that is, let EF be equal to EG: AB shall be equal to CD.

For, the same construction being made, it may be shewn, as before, that AB is double of AF, and CD double of CG, and that the squares on EF, FA are equal to the squares on EG, GC;

but the square on EF is equal to the square on EG, because EF is equal to EG; [*Hypothesis*.

therefore the remaining square on FA is equal to the remaining square on GC, [*Axiom* 3.

and therefore the straight line AF is equal to the straight line CG.

But AB was shewn to be double of AF, and CD double of CG.

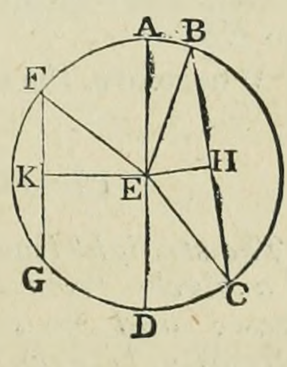
Therefore AB is equal to CD. [*Axiom* 6.

Wherefore, *equal straight lines* &c. Q.E.D.

PROPOSITION 15. *THEOREM.*

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Let $ABCD$ be a circle, of which AD is a diameter, and E the centre; and let BC be nearer to the centre than FG: AD shall be greater than any straight line BC which is not a diameter, and BC shall be greater than FG.



From the centre E draw EH, EK perpendiculars to BC, FG, [I. 12.

and join EB, EC, EF.

Then, because AE is equal to BE, and ED to EC, [I. *Def.* 15.

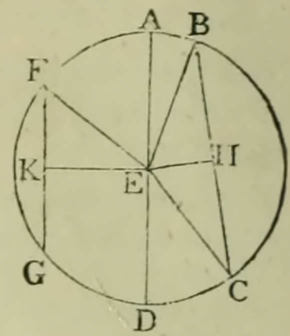
therefore AD is equal to BE, EC; [*Axiom* 2.

but BE, EC are greater than BC; [I. 20.

therefore also AD is greater than BC.

And, because BC is nearer to the centre than FG, [*Hypothesis.*

EH is less than EK. [III. *Def.* 5.



Now it may be shewn, as in the preceding proposition, that BC is double of BH, and FG double of FK, and that the squares on EH, HB are equal to the squares on EK, KF.

But the square on EH is less than the square on EK, because EH is less than EK;

therefore the square on HB is greater than the square on KF;

and therefore the straight line BH is greater than the straight line FK;

and therefore BC is greater than FG.

Next, let BC be greater than FG: BC shall be nearer to the centre than FG, that is, the same construction being made, EH shall be less than EK.

For, because BC is greater than FG, BH is greater than FK.

But the squares on BH, HE are equal to the squares on FK, KE;

and the square on BH is greater than the square on FK, because BH is greater than FK;

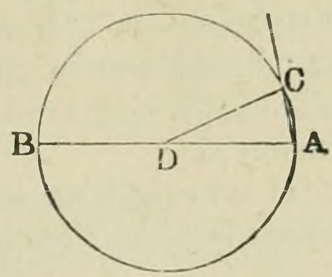
therefore the square on HE is less than the square on KE; and therefore the straight line EH is less than the straight line EK.

Wherefore, *the diameter* &c. Q.E.D.

PROPOSITION 16. *THEOREM.*

The straight line drawn at right angles to the diameter of a circle from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity, between that straight line and the circumference, so as not to cut the circle.

Let ABC be a circle, of which D is the centre and AB a diameter: the straight line drawn at right angles to AB, from its extremity A, shall fall without the circle.



For, if not, let it fall, if possible, within the circle, as AC, and draw DC to the point C, where it meets the circumference.

Then, because DA is equal to DC, [I. *Definition* 15.

the angle DAC is equal to the angle DCA. [I. 5.

But the angle DAC is a right angle; [*Hypothesis.*

therefore the angle DCA is a right angle;

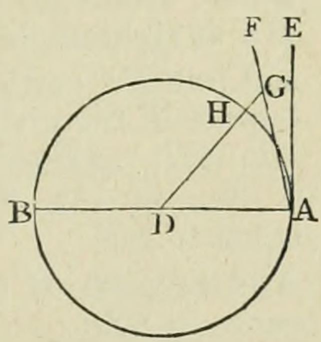
and therefore the angles DAC, DCA are equal to two right angles; which is impossible. [I. 17.

Therefore the straight line drawn from A at right angles to AB does not fall within the circle.

And in the same manner it may be shewn that it does not fall on the circumference.

Therefore it must fall without the circle, as AE.

Also between the straight line AE and the circumference, no straight line can be drawn from the point A, which does not cut the circle.



For, if possible, let AF be between them; and from the centre D draw DG perpendicular to AF; [I. 12.

let DG meet the circumference at H.

Then, because the angle DGA is a right angle, [*Construction.*

the angle DAG is less than a right angle; [I. 17.

therefore DA is greater than DG. [I. 19.

But DA is equal to DH; [I. *Definition* 15.

therefore DH is greater than DG, the less than the greater; which is impossible.

Therefore no straight line can be drawn from the point A between AE and the circumference, so as not to cut the circle.

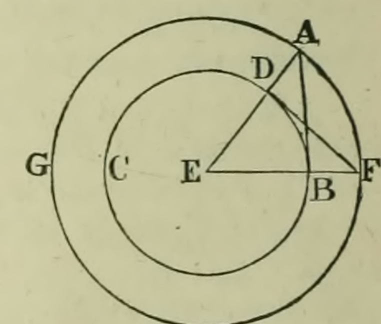
Wherefore, *the straight line* &c. Q.E.D.

Corollary. From this it is manifest, that the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; [III. *Def.* 2.
and that it touches the circle at one point only,
because if it did meet the circle at two points it would fall within it. [III. 2.
Also it is evident, that there can be but one straight line which touches the circle at the same point.

PROPOSITION 17. *PROBLEM.*

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

First, let the given point A be without the given circle BCD: it is required to draw from A a straight line, which shall touch the given circle.



Take E, the centre of the circle, [III. 1.
and join AE cutting the circumference of the given circle at D; and from the centre E, at the distance EA, describe the circle AFG; from the point D draw DF at right angles to EA, [I. 11.
and join EF cutting the circumference of the given circle at B; join AB. AB shall touch the circle BCD.

For, because E is the centre of the circle AFG, EA is equal to EF. [I. *Definition* 15.
And because E is the centre of the circle BCD, EB is equal to ED. [I. *Definition* 15.
Therefore the two sides AE, EB are equal to the two sides FE, ED, each to each;
and the angle at E is common to the two triangles AEB, FED;
therefore the triangle AEB is equal to the triangle FED, and the other angles to the other angles, each to each, to which the equal sides are opposite; [I. 4.

therefore the angle ABE is equal to the angle FDE.
But the angle FDE is a right angle; [*Construction.*
therefore the angle ABE is a right angle. [*Axiom* 1.
And EB is drawn from the centre; but the straight line drawn at right angles to a diameter of a circle, from the extremity of it, touches the circle; [III. 16, *Corollary.*
therefore AB touches the circle.
And AB is drawn from the given point A. Q.E.F.

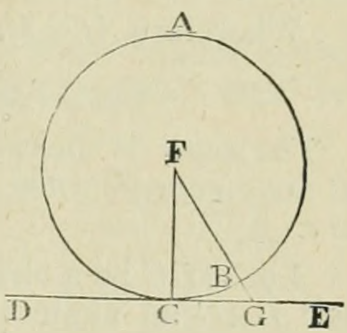
But if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; then DF touches the circle. [III. 16, *Cor.*

PROPOSITION 18. *THEOREM.*

If a straight line touch a circle the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC at the point C; take F, the centre of the circle ABC, and draw the straight line FC: FC shall be perpendicular to DE.

For if not, let FG be drawn from the point F perpendicular to DE, meeting the circumference at B.



Then, because FGC is a right angle, [*Hypothesis.*
FCG is an acute angle; [I. 17.
and the greater angle of every triangle is subtended by the greater side; [I. 19.
therefore FC is greater than FG.
But FC is equal to FB; [I. *Definition* 15.
therefore FB is greater than FG, the less than the greater; which is impossible.
Therefore FG is not perpendicular to DE.

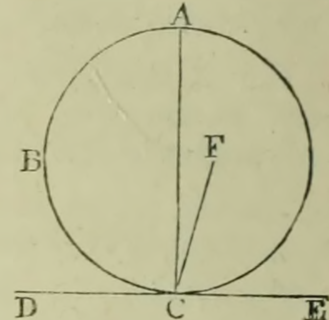
In the same manner it may be shewn that no other straight line from F is perpendicular to DE, but FC; therefore FC is perpendicular to DE.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 19. *THEOREM.*

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC at C, and from C let CA be drawn at right angles to DE: the centre of the circle shall be in CA.



For, if not, if possible, let F be the centre, and join CF.

Then, because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular to DE; [III. 18.

therefore the angle FCE is a right angle.

But the angle ACE is also a right angle; [*Construction.*

therefore the angle FCE is equal to the angle ACE, [*Ax.* 11.

the less to the greater; which is impossible.

Therefore F is not the centre of the circle ABC.

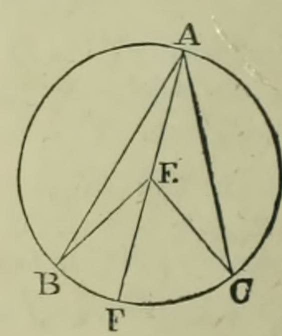
In the same manner it may be shewn that no other point out of CA is the centre; therefore the centre is in CA.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 20. *THEOREM.*

The angle at the centre of a circle is double of the angle at the circumference on the same base, that is, on the same arc.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same arc, BC, for their base: the angle BEC shall be double of the angle BAC.



Join AE, and produce it to F.

First let the centre of the circle be within the angle BAC.

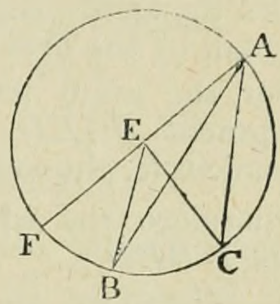
Then, because EA is equal to EB, the angle EAB is equal to the angle EBA; [I. 5.

therefore the angles EAB, EBA are double of the angle EAB.

But the angle BEF is equal to the angles EAB, EBA; [I. 32.
therefore the angle BEF is double of the angle EAB.

For the same reason the angle FEC is double of the angle EAC.

Therefore the whole angle BEC is double of the whole angle BAC.



Next, let the centre of the circle be without the angle BAC.

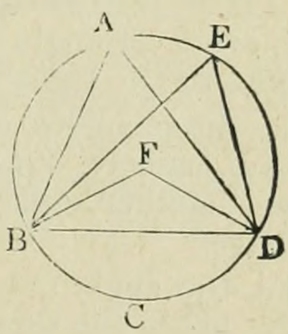
Then it may be shewn, as in the first case, that the angle FEC is double of the angle FAC, and that the angle FEB, a part of the first, is double of the angle FAB, a part of the other; therefore the remaining angle BEC is double of the remaining angle BAC.

Wherefore, *the angle at the centre* &c. Q.E.D.

PROPOSITION 21. *THEOREM.*

The angles in the same segment of a circle are equal to one another.

Let $ABCD$ be a circle, and BAD, BED angles in the same segment $BAED$: the angles BAD, BED shall be equal to one another.



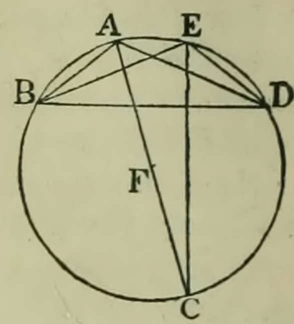
Take F the centre of the circle $ABCD$. [III. 1.

First let the segment $BAED$ be greater than a semicircle.

Join BF, DF.

Then, because the angle BFD is at the centre, and the angle BAD is at the circumference, and that they have the same arc for their base, namely, BCD;
therefore the angle BFD is double of the angle BAD. [III. 20.
For the same reason the angle BFD is double of the angle BED.
Therefore the angle BAD is equal to the angle BED. [*Ax.* 7.

Next, let the segment $BAED$ be not greater than a semicircle.



Draw AF to the centre, and produce it to meet the circumference at C, and join CE.

Then the segment $BAEC$ is greater than a semicircle, and therefore the angles BAC, BEC in it, are equal, by the first case.

For the same reason, because the segment $CAED$ is greater than a semicircle, the angles CAD, CED are equal.

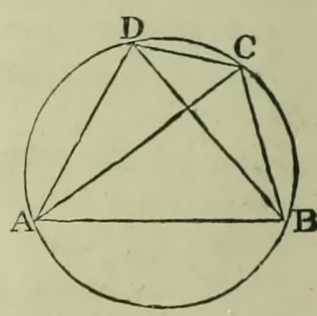
Therefore the whole angle BAD is equal to the whole angle BED. [*Axiom* 2.

Wherefore, *the angles in the same segment* &c. Q.E.D.

PROPOSITION 22. *THEOREM.*

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let $ABCD$ be a quadrilateral figure inscribed in the circle $ABCD$: any two of its opposite angles shall be together equal to two right angles.



Join AC, BD.

Then, because the three angles of every triangle are together equal to two right angles, [I. 32.

the three angles of the triangle CAB, namely, CAB, ACB, ABC are together equal to two right angles.

But the angle CAB is equal to the angle CDB, because they are in the same segment $CDAB$; [III. 21.

and the angle ACB is equal to the angle ADB, because they are in the same segment $ADCB$;

therefore the two angles CAB, ACB are together equal to the whole angle ADC. [*Axiom* 2.

To each of these equals add the angle ABC;

therefore the three angles CAB, ACB, ABC, are equal to the two angles ABC, ADC.

But the angles CAB, ACB, ABC are together equal to two right angles; [I. 32.

therefore also the angles ABC, ADC are together equal to two right angles.

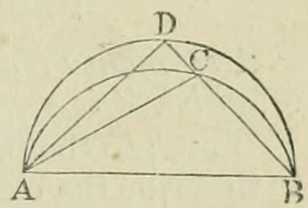
In the same manner it may be shewn that the angles BAD, BCD are together equal to two right angles.

Wherefore, *the opposite angles* &c. Q.E.D.

PROPOSITION 23. *THEOREM.*

On the same straight line, and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, on the same straight line AB, and on the same side of it, let there be two similar segments of circles ACB, ADB, not coinciding with one another.



Then, because the circle ACB cuts the circle ADB at the two points A, B, they cannot cut one another at any other point; [III. 10.

therefore one of the segments must fall within the other; let ACB fall within ADB; draw the straight line BCD, and join AC, AD.

Then, because ACB, ADB are, by hypothesis, similar segments of circles, and that similar segments of circles contain equal angles, [III. *Definition* 11.

therefore the angle ACB is equal to the angle ADB;

that is, the exterior angle of the triangle ACD is equal to the interior and opposite angle;

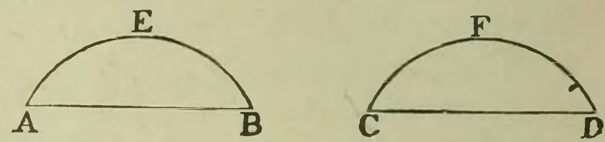
which is impossible. [I. 16.

Wherefore, *on the same straight line* &c. Q.E.D.

PROPOSITION 24. *THEOREM.*

Similar segments of circles on equal straight lines are equal to one another.

Let *AEB*, *CFD* be similar segments of circles on the equal straight lines *AB*, *CD*: the segment *AEB* shall be equal to the segment *CFD*.



For if the segment *AEB* be applied to the segment *CFD*, so that the point *A* may be on the point *C*, and the straight line *AB* on the straight line *CD*, the point *B* will coincide with the point *D*, because *AB* is equal to *CD*.

Therefore, the straight line *AB* coinciding with the straight line *CD*, the segment *AEB* must coincide with the segment *CFD*; [III. 23.

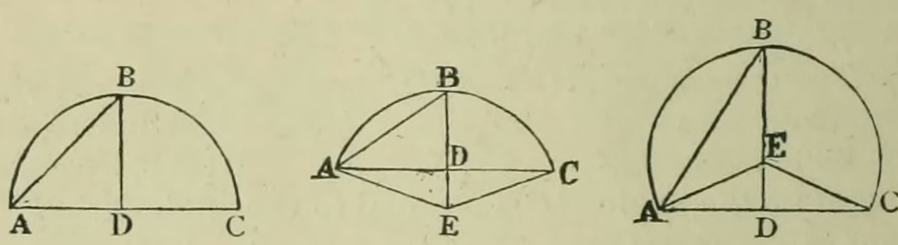
and is therefore equal to it.

Wherefore, *similar segments* &c. Q.E.D.

PROPOSITION 25. *PROBLEM.*

A segment of a circle being given, to describe the circle of which it is a segment.

Let *ABC* be the given segment of a circle: it is required to describe the circle of which it is a segment.



Bisect *AC* at *D*; [I. 10.

from the point *D* draw *DB* at right angles to *AC*; [I. 11.

and join *AB*.

First, let the angles *ABD*, *BAD*, be equal to one another.

Then *DB* is equal to *DA*; [I. 6.

but *DA* is equal to *DC*; [*Construction.*

therefore *DB* is equal to *DC*. [*Axiom* 1.

Therefore the three straight lines DA, DB, DC are all equal; and therefore D is the centre of the circle. [III. 9.

From the centre D, at the distance of any of the three DA, DB, DC, describe a circle; this will pass through the other points, and the circle of which ABC is a segment is described.

And because the centre D is in AC, the segment ABC is a semicircle.

Next, let the angles ABD, BAD be not equal to one another.

At the point A, in the straight line AB, make the angle BAE equal to the angle ABD; [I. 23.

produce BD, if necessary, to meet AE at E, and join EC.

Then, because the angle BAE is equal to the angle ABE, [*Construction.*

EA is equal to EB. [I. 6.

And because AD is equal to CD, [*Construction.*

and DE is common to the two triangles ADE, CDE;

the two sides AD, DE are equal to the two sides CD, DE, each to each;

and the angle ADE is equal to the angle CDE, for each of them is a right angle; [*Construction.*

therefore the base EA is equal to the base EC. [I. 4.

But EA was shewn to be equal to EB;

therefore EB is equal to EC. [*Axiom* 1.

Therefore the three straight lines EA, EB, EC are all equal;

and therefore E is the centre of the circle. [III. 9.

From the centre E, at the distance of any of the three EA, EB, EC, describe a circle; this will pass through the other points, and the circle of which ABC is a segment is described.

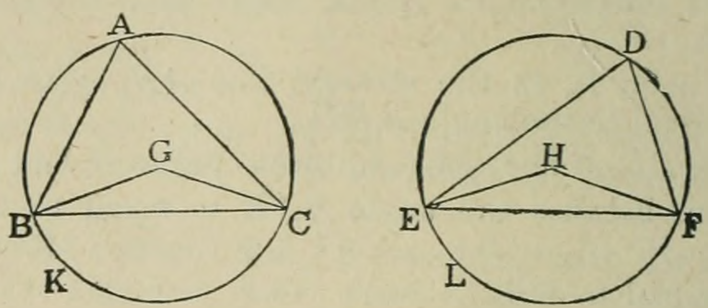
And it is evident, that if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC, which is therefore less than a semicircle; but if the angle ABD be less than the angle BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle.

Wherefore, *a segment of a circle being given, the circle has been described of which it is a segment.* Q.E.F.

PROPOSITION 26. *THEOREM.*

In equal circles, equal angles stand on equal arcs, whether they be at the centres or circumferences.

Let *ABC*, *DEF* be equal circles; and let *BGC*, *EHF* be equal angles in them at their centres, and *BAC*, *EDF* equal angles at their circumferences: the arc *BKC* shall be equal to the arc *ELF*.



Join *BC*, *EF*.

Then, because the circles *ABC*, *DEF* are equal, [*Hyp.*
the straight lines from their centres are equal; [III. *Def.* 1.
therefore the two sides *BG*, *GC* are equal to the two sides *EH*, *HF*, each to each;
and the angle at *G* is equal to the angle at *H*; [*Hypothesis.*
therefore the base *BC* is equal to the base *EF*. [I. 4.

And because the angle at *A* is equal to the angle at *D*, [*Hyp.*
the segment *BAC* is similar to the segment *EDF*; [III. *Def.* 11.
and they are on equal straight lines *BC*, *EF*.
But similar segments of circles on equal straight lines are equal to one another; [III. 24.
therefore the segment *BAC* is equal to the segment *EDF*.

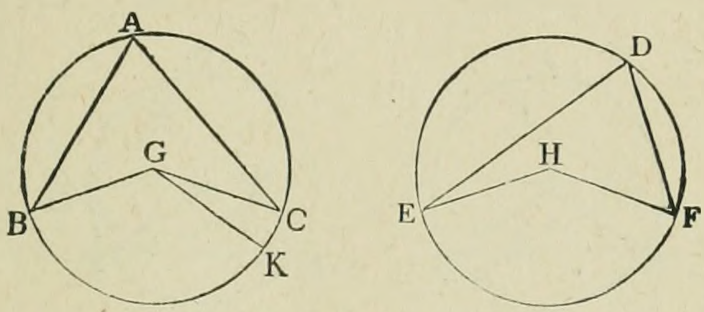
But the whole circle *ABC* is equal to the whole circle *DEF*; [*Hypothesis.*
therefore the remaining segment *BKC* is equal to the remaining segment *ELF*; [*Axiom* 3.
therefore the arc *BKC* is equal to the arc *ELF*.

Wherefore, *in equal circles* &c. Q.E.D.

PROPOSITION 27. *THEOREM.*

In equal circles, the angles which stand on equal arcs are equal to one another, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF at their centres, and the angles BAC, EDF at their circumferences, stand on equal arcs BC, EF: the angle BGC shall be equal to the angle EHF, and the angle BAC equal to the angle EDF.



If the angle BGC be equal to the angle EHF, it is manifest that the angle BAC is also equal to the angle EDF. [III. 20, *Axiom* 7.

But, if not, one of them must be the greater. Let BGC be the greater, and at the point G, in the straight line BG, make the angle BGK equal to the angle EHF. [I. 23.

Then, because the angle BGK is equal to the angle EHF, and that in equal circles equal angles stand on equal arcs, when they are at the centres, [III. 26.

therefore the arc BK is equal to the arc EF.

But the arc EF is equal to the arc BC; [*Hypothesis.*

therefore the arc BK is equal to the arc BC, [*Axiom* 1.

the less to the greater; which is impossible.

Therefore the angle BGC is not unequal to the angle EHF, that is, it is equal to it.

And the angle at A is half of the angle BGC, and the angle at D is half of the angle EHF; [III. 20.

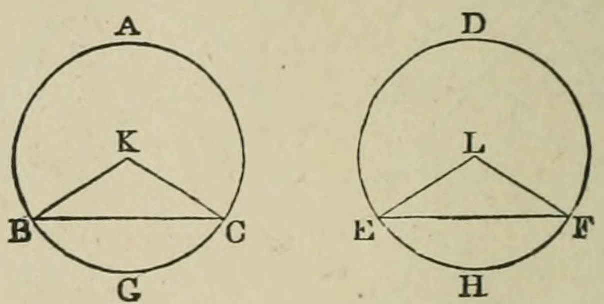
therefore the angle at A is equal to the angle at D. [*Ax.* 7.

Wherefore, *in equal circles* &c. Q.E.D.

PROPOSITION 28. *THEOREM.*

In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater, and the less equal to the less.

Let ABC, DEF be equal circles, and BC, EF equal straight lines in them, which cut off the two greater arcs BAC, EDF, and the two less arcs BGC, EHF: the greater arc BAC shall be equal to the greater arc EDF, and the less arc BGC equal to the less arc EHF.



Take K, L, the centres of the circles, [III. 1.
and join BK, KC, EL, LF.

Then, because the circles are equal, [*Hypothesis.*
the straight lines from their centres are equal; [III. *Def.* 1.
therefore the two sides BK, KC are equal to the two sides EL, LF, each to each;
and the base BC is equal to the base EF; [*Hypothesis.*
therefore the angle BKC is equal to the angle ELF. [I. 8.
But in equal circles equal angles stand on equal arcs, when they are at the centres, [III. 26.
therefore the arc BGC is equal to the arc EHF.

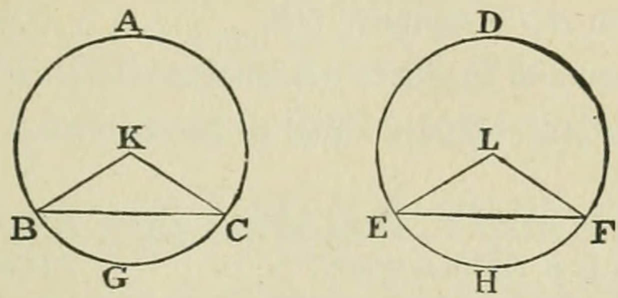
But the circumference $ABGC$ is equal to the circumference $DEHF$; [*Hypothesis.*
therefore the remaining arc BAC is equal to the remaining arc EDF. [*Axiom* 3.

Wherefore, *in equal circles* &c. Q.E.D.

PROPOSITION 29. *THEOREM.*

In equal circles, equal arcs are subtended by equal straight lines.

Let *ABC*, *DEF* be equal circles, and let *BGC*, *EHF* be equal arcs in them, and join *BC*, *EF*: the straight line *BC* shall be equal to the straight line *EF*.



Take *K*, *L*, the centres of the circles, [III. 1.
and join *BK*, *KC*, *EL*, *LF*.

Then, because the arc *BGC* is equal to the arc *EHF*, [*Hypothesis.*
the angle *BKC* is equal to the angle *ELF*. [III. 27.
And because the circles *ABC*, *DEF* are equal, [*Hypothesis.*
the straight lines from their centres are equal; [III. *Def.* 1.
therefore the two sides *BK*, *KC* are equal to the two sides *EL*, *LF*, each to each;
and they contain equal angles;
therefore the base *BC* is equal to the base *EF*. [I. 4.

Wherefore, *in equal circles* &c. Q.E.D.

PROPOSITION 30. *PROBLEM.*

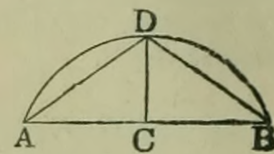
To bisect a given arc, that is, to divide it into two equal parts.

Let ADB be the given arc: it is required to bisect it.

Join AB;

bisect it at C; [I. 10.

from the point C draw CD at right angles to AB meeting the arc at D. [I. 11.



The arc ADB shall be bisected at the point D.

Join AD, DB.

Then, because AC is equal to CB, [*Construction.*

and CD is common to the two triangles ACD, BCD;

the two sides AC, CD are equal to the two sides BC, CD, each to each;

and the angle ACD is equal to the angle BCD, because each of them is a right angle; [*Construction.*

therefore the base AD is equal to the base BD. [I. 4.

But equal straight lines cut off equal arcs, the greater equal to the greater, and the less equal to the less; [III. 28.

and each of the arcs AD, DB is less than a semi-circumference, because DC, if produced, is a diameter; [III. 1. *Cor*

therefore the arc AD is equal to the arc DB.

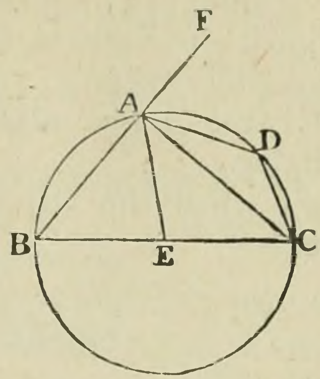
Wherefore *the given arc is bisected at D.* Q.E.F.

PROPOSITION 31. *THEOREM.*

In a circle the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let $ABCD$ be a circle, of which BC is a diameter and E the centre; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC: the angle in the semicircle BAC shall be a right angle; but the angle in the segment ABC, which is greater than a

semicircle, shall be less than a right angle; and the angle in the segment ADC, which is less than a semicircle, shall be greater than a right angle.



Join AE, and produce BA to F.

Then, because EA is equal to EB, [I. *Definition* 15.

the angle EAB is equal to the angle EBA; [I. 5.

and, because EA is equal to EC,
the angle EAC is equal to the angle ECA;

therefore the whole angle BAC is equal to the two angles, ABC, ACB. [*Axiom* 2.

But FAC, the exterior angle of the triangle ABC, is equal to the two angles ABC, ACB; [I. 32.

therefore the angle BAC is equal to the angle FAC, [*Ax.* 1.

and therefore each of them is a right angle. [I. *Def.* 10.

Therefore the angle in a semicircle BAC is a right angle.

And because the two angles ABC, BAC, of the triangle ABC, are together less than two right angles, [I. 17.

and that BAC has been shewn to be a right angle,

therefore the angle ABC is less than a right angle.

Therefore the angle in a segment ABC, greater than a semicircle, is less than a right angle.

And because $ABCD$ is a quadrilateral figure in a circle, any two of its opposite angles are together equal to two right angles; [III. 22.

therefore the angles ABC, ADC are together equal to two right angles.

But the angle ABC has been shewn to be less than a right angle;

therefore the angle ADC is greater than a right angle.

Therefore the angle in a segment ADC, less than a semicircle, is greater than a right angle.

Wherefore, *the angle* &c. Q.E.D.

COROLLARY. From the demonstration it is manifest that if one angle of a triangle be equal to the other two, it is a right angle.

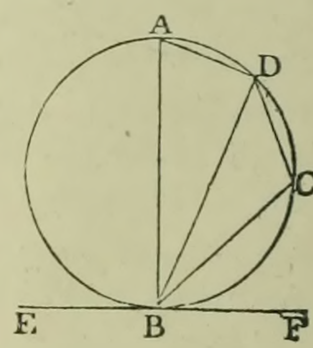
For the angle adjacent to it is equal to the same two angles; [I. 32.

and when the adjacent angles are equal, they are right angles. [I. *Definition* 10.

PROPOSITION 32. *THEOREM.*

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

Let the straight line EF touch the circle $ABCD$ at the point B, and from the point B let the straight line BD be drawn, cutting the circle: the angles which BD makes with the touching line EF, shall be equal to the angles in the alternate segments of the circle; that is, the angle DBF shall be equal to the angle in the segment BAD, and the angle DBE shall be equal to the angle in the segment BCD.



From the point B draw BA at right angles to EF, [I. 11.

and take any point C in the arc BD, and join AD, DC, CB.

Then, because the straight line EF touches the circle $ABCD$ at the point B, [*Hyp.*

and BA is drawn at right angles to the touching line from the point of contact B, [*Construction.*

therefore the centre of the circle is in BA. [III. 19.

Therefore the angle ADB, being in a semicircle, is a right angle. [III. 31.

Therefore the other two angles BAD, ABD are equal to a right angle. [I. 32.

But ABF is also a right angle. [*Construction.*

Therefore the angle ABF is equal to the angles BAD, ABD.

From each of these equals take away the common angle ABD;

therefore the remaining angle DBF is equal to the remaining angle BAD, [*Axiom* 3

which is in the alternate segment of the circle.

And because $ABCD$ is a quadrilateral figure in a circle, the opposite angles BAD, BCD are together equal to two right angles. [III. 22.

But the angles DBF, DBE are together equal to two right angles. [I. 13.

Therefore the angles DBF, DBE are together equal to the angles BAD, BCD.

And the angle DBF has been shewn equal to the angle BAD;

therefore the remaining angle DBE is equal to the remaining angle BCD, [*Axiom* 3.

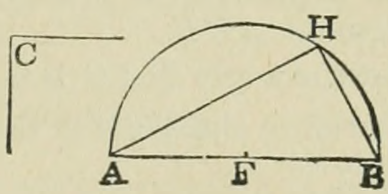
which is in the alternate segment of the circle.

Wherefore, *if a straight line* &c. Q.E.D.

PROPOSITION 33. *PROBLEM.*

On a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and C the given rectilineal angle: it is required to describe, on the given straight line AB, a segment of a circle containing an angle equal to the angle C.

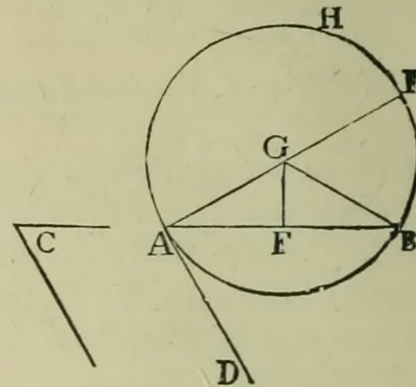


First, let the angle C be a right angle.

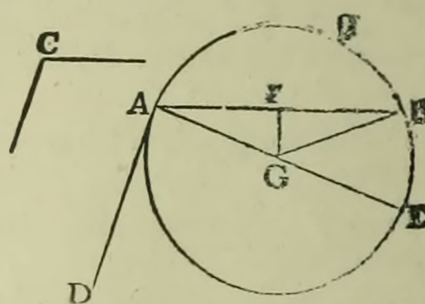
Bisect AB at F, [I. 10.

and from the centre F, at the distance FB, describe the semicircle AHB.

Then the angle AHB in a semicircle is equal to the right angle C. [III. 31



But if the angle C be not a right angle, at the point A, in the straight line AB, make the angle BAD equal to the angle C; [I. 23.
from the point A, draw AE at right angles to AD; [I. 11.
bisect AB at F; [I. 10.
from the point F, draw FG at right angles to AB; [I. 11.
and join GB.



Then, because AF is equal to BF, [*Const.*
and FG is common to the two triangles AFG, BFG;
the two sides AF, FG are equal to the two sides BF, FG, each to each;
and the angle AFG is equal to the angle BFG; [I. *Definition* 10.
therefore the base AG is equal to the base BG; [I. 4.
and therefore the circle described from the centre G, at the distance GA, will pass through the point B.

Let this circle be described; and let it be AHB.

The segment AHB shall contain an angle equal to the given rectilineal angle C.

Because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, [*Construction.*
therefore AD touches the circle. [III. 16. *Corollary.*
And because AB is drawn from the point of contact A, the angle DAB is equal to the angle in the alternate segment AHB. [III. 32.
But the angle DAB is equal to the angle C. [*Constr.*
Therefore the angle in the segment AHB is equal to the angle C. [*Axiom* 1.

Wherefore, *on the given straight line AB, the segment AHB of a circle has been described, containing an angle equal to the given angle C.* Q.E.F.

PROPOSITION 34. *PROBLEM.*

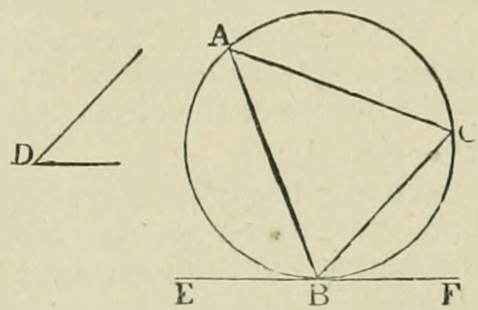
From a given circle to cut off a segment containing an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle: it is required to cut off from the circle ABC a segment containing an angle equal to the angle D.

Draw the straight line EF touching the circle ABC at the point B; [III. 17.

and at the point B, in the straight line BF, make the angle FBC equal to the angle D. [I. 23.

The segment BAC shall contain an angle equal to the angle D.



Because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, [*Constr.*

therefore the angle FBC is equal to the angle in the alternate segment BAC of the circle. [III. 32.

But the angle FBC is equal to the angle D. [*Construction.*

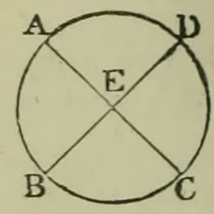
Therefore the angle in the segment BAC is equal to the angle D. [*Axiom* 1.

Wherefore, *from the given circle ABC, the segment BAC has been cut off, containing an angle equal to the given angle D.* Q.E.F.

PROPOSITION 35. *THEOREM.*

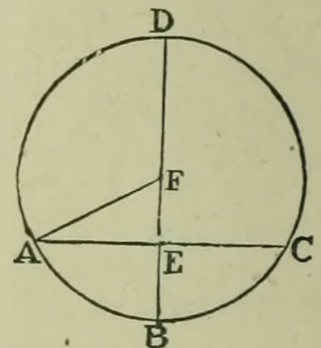
If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD cut one another at the point E, within the circle $ABCD$: the rectangle contained by AE, EC shall be equal to the rectangle contained by BE, ED.



If AC and BD both pass through the centre, so that E is the centre, it is evident, since EA, EB, EC, ED are all equal, that the rectangle AE, EC is equal to the rectangle BE, ED.

But let one of them, BD, pass through the centre, and cut the other AC, which does not pass through the centre, at right angles, at the point E. Then, if BD be bisected at F, F is the centre of the circle $ABCD$; join AF.



Then, because the straight line BD which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles at the point E, [*Hypothesis.*
AE is equal to EC. [III. 3.

And because the straight line BD is divided into two equal parts at the point F, and into two unequal parts at the point E, the rectangle BE, ED, together with the square on EF, is equal to the square on FB, [II. 5.
that is, to the square on AF.

But the square on AF is equal to the squares on AE, EF. [I. 47.

Therefore the rectangle BE, ED, together with the square on EF, is equal to the squares on AE, EF. [*Axiom* 1.

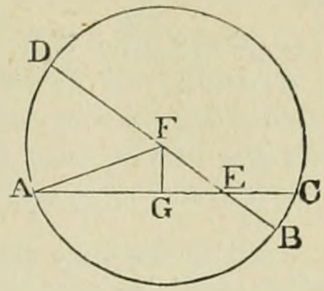
Take away the common square on EF;

then the remaining rectangle BE, ED, is equal to the remaining square on AE,

that is, to the rectangle AE, EC.

Next, let BD, which passes through the centre, cut the other AC, which does not pass through the centre, at the point E, but not at right angles. Then, if BD be bisected at F, F is the centre of the circle $ABCD$; join AF, and from F draw FG perpendicular to AC. [I. 12.

Then AG is equal to GC; [III. 3.
therefore the rectangle AE, EC, together with the square on EG, is equal to the square on AG. [II. 5.
To each of these equals add the square on GF;
then the rectangle AE, EC, together with the squares on EG, GF, is equal to the squares on AG, GF. [*Axiom* 2.



But the squares on EG, GF are equal to the square on EF;

and the squares on AG, GF are equal to the square on AF. [I. 47.

Therefore the rectangle AE, EC, together with the square on EF, is equal to the square on AF,

that is, to the square on FB.

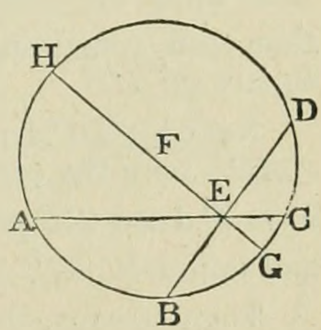
But the square on FB is equal to the rectangle BE, ED, together with the square on EF. [II. 5.
Therefore the rectangle AE, EC, together with the square on EF, is equal to the rectangle BE, ED, together with the square on EF.

Take away the common square on EF;

then the remaining rectangle AE, EC is equal to the remaining rectangle BE, ED. [*Axiom* 3.

Lastly, let neither of the straight lines AC, BD pass through the centre.
Take the centre F, [III. 1.
and through E, the intersection of the straight lines AC, BD, draw the diameter $GEFH$.

Then, as has been shewn, the rectangle GE, EH is equal to the rectangle AE, EC, and also to the rectangle BE, ED;
therefore the rectangle AE, EC is equal to the rectangle BE, ED. [*Axiom* 1.

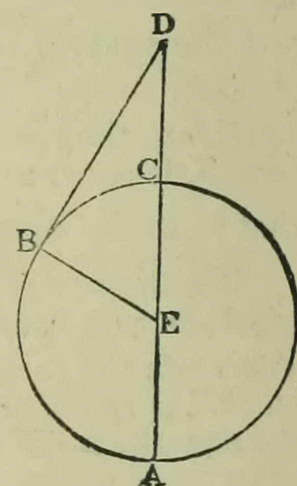


Wherefore, *if two straight lines* &c. Q.E.D.

PROPOSITION 36. *THEOREM.*

If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.

Let D be any point without the circle ABC, and let DCA, DB be two straight lines drawn from it, of which DCA cuts the circle and DB touches it: the rectangle AD, DC shall be equal to the square on DB.



First, let DCA pass through the centre E, and join EB.

Then EBD is a right angle. [III. 18.

And because the straight line AC is bisected at E, and produced to D, the rectangle AD, DC together with the square on EC is equal to the square on ED. [II. 6.

But EC is equal to EB; therefore the rectangle AD, DC together with the square on EB is equal to the square on ED.

But the square on ED is equal to the squares on EB, BD, because EBD is a right angle. [I. 47.

Therefore the rectangle AD, DC, together with the square on EB is equal to the squares on EB, BD.

Take away the common square on EB;

then the remaining rectangle AD, DC is equal to the square on DB. [*Axiom* 3.

Next let DCA not pass through the centre of the circle ABC; take the centre E; [III. 1.

from E draw EF perpendicular to AC; [I. 12.

and join EB, EC, ED.

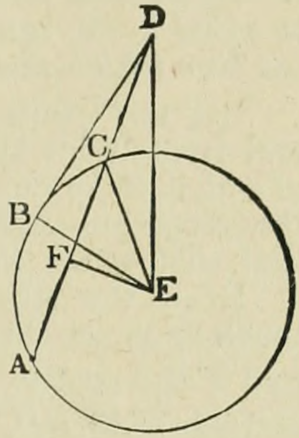
Then, because the straight line EF which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it also bisects it; [III. 3.

therefore AF is equal to FC.

And because the straight line AC is bisected at F, and produced to D, the rectangle AD, DC, together with the square on FC, is equal to the square on FD. [II. 6.

To each of these equals add the square on FE.

Therefore the rectangle AD, DC together with the squares on CF, FE, is equal to the squares on DF, FE. [*Axiom* 2.



But the squares on CF, FE are equal to the square on CE, because CFE is a right angle; [I. 47.

and the squares on DF, FE are equal to the square on DE.

Therefore the rectangle AD, DC, together with the square on CE, is equal to the square on DE.

But CE is equal to BE;

therefore the rectangle AD, DC, together with the square on BE, is equal to the square on DE.

But the square on DE is equal to the squares on DB, BE, because EBD is a right angle. [I. 47.

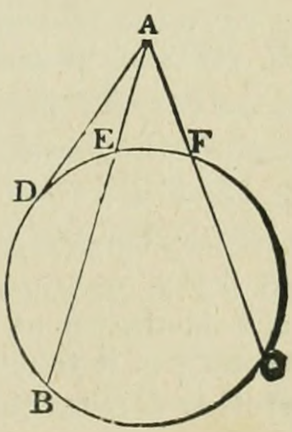
Therefore the rectangle AD, DC, together with the square on BE, is equal to the squares on DB, BE.

Take away the common square on BE;

then the remaining rectangle AD, DC is equal to the square on DB. [*Axiom* 3.

Wherefore, *if from any point* &c. Q.E.D.

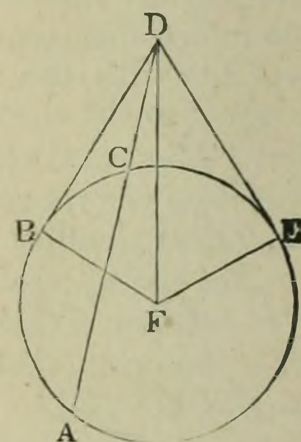
Corollary. If from any point without a circle, there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circles are equal to one another; namely, the rectangle BA, AE is equal to the rectangle CA, AF; for each of them is equal to the square on the straight line AD, which touches the circle.



PROPOSITION 37. *THEOREM.*

If from any point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets the circle, the line which meets the circle shall touch it.

Let any point D be taken without the circle ABC, and from it let two straight lines DCA, DB be drawn, of which DCA cuts the circle, and DB meets it; and let the rectangle AD, DC be equal to the square on DB. DB shall touch the circle.



Draw the straight line DE, touching the circle ABC; [III. 17.
find F the centre, [III. 1.
and join FB, FD, FE.

Then the angle FED is a right angle. [III. 18.
And because DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal to the square on DE. [III. 36.
But the rectangle AD, DC is equal to the square on DB. [*Hyp.*
Therefore the square on DE is equal to the square on DB; [*Ax.* 1.
therefore the straight line DE is equal to the straight line DB.

And EF is equal to BF; [I. *Definition* 15.
therefore the two sides DE, EF are equal to the two sides DB, BF each to each;
and the base DF is common to the two triangles DEF, DBF;
therefore the angle DEF is equal to the angle DBF. [I. 8.
But DEF is a right angle; [*Construction.*
therefore also DBF is a right angle.
And BF, if produced, is a diameter; and the straight line which is drawn at right angles to a diameter from the extremity of it touches the circle; [III. 16. *Corollary.*
therefore DB touches the circle ABC.

Wherefore, *if from a point* &c. Q.E.D.

EXERCISES IN EUCLID.

EXERCISES IN EUCLID.

I. 1 to 15.

1. On a given straight line describe an isosceles triangle having each of the sides equal to a given straight line.

2. In the figure of I. 2 if the diameter of the smaller circle is the radius of the larger, shew where the given point and the vertex of the constructed triangle will be situated.

3. If two straight lines bisect each other at right angles, any point in either of them is equidistant from the extremities of the other.

4. If the angles ABC and ACB at the base of an isosceles triangle be bisected by the straight lines BD, CD, shew that DBC will be an isosceles triangle.

5. BAC is a triangle having the angle B double of the angle A. If BD bisects the angle B and meets AC at D, shew that BD is equal to AD.

6. In the figure of I. 5 if FC and BG meet at H shew that FH and GH are equal.

7. In the figure of I. 5 if FC and BG meet at H, shew that AH bisects the angle BAC.

8. The sides AB, AD of a quadrilateral $ABCD$ are equal, and the diagonal AC bisects the angle BAD: shew that the sides CB and CD are equal, and that the diagonal AC bisects the angle BCD.

9. ACB, ADB are two triangles on the same side of AB, such that AC is equal to BD, and AD is equal to BC, and AD and BC intersect at O: shew that the triangle AOB is isosceles.

10. The opposite angles of a rhombus are equal.

11. A diagonal of a rhombus bisects each of the angles through which it passes.

12. If two isosceles triangles are on the same base the straight line joining their vertices, or that straight line produced, will bisect the base at right angles.

13. Find a point in a given straight line such that its distances from two given points may be equal.

14. Through two given points on opposite sides of a given straight line draw two straight lines which shall meet in that given straight line, and include an angle bisected by that given straight line.

15. A given angle BAC is bisected; if CA is produced to G and the angle BAG bisected, the two bisecting lines are at right angles.

16. If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line.

I. 16 to 26.

17. ABC is a triangle and the angle A is bisected by a straight line which meets BC at D; shew that BA is greater than BD, and CA greater than CD.

18. In the figure of I. 17 shew that ABC and ACB are together less than two right angles, by joining A to any point in BC.

19. $ABCD$ is a quadrilateral of which AD is the longest side and BC the shortest: shew that the angle ABC is greater than the angle ADC, and the angle BCD greater than the angle BAD.

20. If a straight line be drawn through A one of the angular points of a square, cutting one of the opposite sides, and meeting the other produced at F, shew that AF is greater than the diagonal of the square.

21. The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two, equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.

22. The sum of the distances of any point from the three angles of a triangle is greater than half the sum of the sides of the triangle.

23. The four sides of any quadrilateral are together greater than the two diagonals together.

24. The two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the base.

25. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

26. If the angle C of a triangle is equal to the sum of the angles A and B, the side AB is equal to twice the straight line joining C to the middle point of AB.

27. Construct a triangle, having given the base. one of the angles at the base, and the sum of the sides.

28. The perpendiculars let fall on two sides of a triangle from any point in the straight line bisecting the angle between them are equal to each other.

29. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines shall be equal.

30. Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

31. A straight line bisects the angle A of a triangle ABC; from B a perpendicular is drawn to this bisecting straight line, meeting it at D, and BD is produced to meet AC or AC produced at E: shew that BD is equal to DE.

32. AB, AC are any two straight lines meeting at A: through any point P draw a straight line meeting them at E and F, such that AE may be equal to AF.

33. Two right-angled triangles have their hypotenuses equal, and a side of one equal to a side of the other: shew that they are equal in all respects.

I. 27 to 31.

34. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

35. If two straight lines A and B are respectively parallel to two others C and D, shew that the inclination of A to B is equal to that of C to D.

36. A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is

drawn and terminated by the parallel straight lines. Shew that the second straight line is bisected at the middle point of the first.

37. If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel straight lines.

38. If the straight line bisecting the exterior angle of a triangle be parallel to the base, shew that the triangle is isosceles.

39. Find a point B in a given straight line CD, such that if AB be drawn to B from a given point A, the angle ABC will be equal to a given angle.

40. If a straight line be drawn bisecting one of the angles of a triangle to meet the opposite side, the straight lines drawn from the point of section parallel to the other sides, and terminated by these sides, will be equal.

41. The side BC of a triangle ABC is produced to a point D; the angle ACB is bisected by the straight line CE which meets AB at E. A straight line is drawn through E parallel to BC, meeting AC at F, and the straight line bisecting the exterior angle ACD at G. Shew that EF is equal to FG.

42. AB is the hypotenuse of a right-angled triangle ABC: find a point D in AB such that DB may be equal to the perpendicular from D on AC.

43. ABC is an isosceles triangle: find points D, E in the equal sides AB, AC such that BD, DE, EC may all be equal.

44. A straight line drawn at right angles to BC the base of an isosceles triangle ABC cuts the side AB at D and CA produced at E: shew that AED is an isosceles triangle.

I. 32.

45. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides; shew that the angles made by them with the base are each equal to half the vertical angle.

46. On the sides of any triangle ABC equilateral triangles BCD, CAE, ABF are described, all external: shew that the straight lines AD, BE, CF are all equal.

47. What is the magnitude of an angle of a regular octagon?

48. Through two given points draw two straight lines forming with a straight line given in position an equilateral triangle.

49. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

50. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA; and DC is drawn: shew that BCD is a right angle.

51. ABC is a triangle, and the exterior angles at B and C are bisected by the straight lines BD, CD respectively, meeting at D: shew that the angle BDC together with half the angle BAC make up a right angle.

52. Shew that any angle of a triangle is obtuse, right, or acute, according as it is greater than, equal to, or less than the other two angles of the triangle taken together.

53. Construct an isosceles triangle having the vertical angle four times each of the angles at the base.

54. In the triangle ABC the side BC is bisected at E and AB at G; AE is produced to F so that EF is equal to AE, and CG is produced to H so that GH is equal to CG: shew that FB and HB are in one straight line.

55. Construct an isosceles triangle which shall have one-third of each angle at the base equal to half the vertical angle.

56. AB, AC are two straight lines given in position: it is required to find in them two points P and Q, such that, PQ being joined, AP and PQ may together be equal to a given straight line, and may contain an angle equal to a given angle.

57. Straight lines are drawn through the extremities of the base of an isosceles triangle, making angles with it on the side remote from the vertex, each equal to one-third of one of the equal angles of the triangle and meeting the sides produced: shew that three of the triangles thus formed are isosceles.

58. AEB, CED are two straight lines intersecting at E; straight lines AC, DB are drawn forming two triangles ACE, BED; the angles ACE, DBE are bisected by the straight lines CF, BF, meeting at F. Shew that the angle CFB is equal to half the sum of the angles EAC, EDB.

59. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

60. From the angle A of a triangle ABC a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at D; from the angle B a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at E: shew that the straight lines which join D and E to the middle point of AB are equal.

61. From the angles at the base of a triangle perpendiculars are drawn to the opposite sides, produced if necessary: shew that the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the middle point of the base.

62. In the figure of I. 1, if C and H be the points of intersection of the circles, and AB be produced to meet one of the circles at K, shew that CHK is an equilateral triangle.

63. The straight lines bisecting the angles at the base of an isosceles triangle meet the sides at D and E: shew that DE is parallel to the base.

64. AB, AC are two given straight lines, and P is a given point in the former: it is required to draw through P a straight line to meet AC at Q, so that the angle APQ may be three times the angle AQP.

65. Construct a right-angled triangle, having given the hypotenuse and the sum of the sides.

66. Construct a right-angled triangle, having given the hypotenuse and the difference of the sides.

67. Construct a right-angled triangle, having given the hypotenuse and the perpendicular from the right angle on it.

68. Construct a right-angled triangle, having given the perimeter and an angle.

69. Trisect a right angle.

70. Trisect a given finite straight line.

71. From a given point it is required to draw to two parallel straight lines, two equal straight lines at right angles to each other.

72. Describe a triangle of given perimeter, having its angles equal to those of a given triangle.

I. 33, 34.

73. If a quadrilateral have two of its opposite sides parallel, and the two others equal but not parallel, any two of its opposite angles are together equal to two right angles.

74. If a straight line which joins the extremities of two equal straight lines, not parallel, make the angles on the same side of it equal to each other, the straight line which joins the other extremities will be parallel to the first.

75. No two straight lines drawn from the extremities of the base of a triangle to the opposite sides can possibly bisect each other.

76. If the opposite sides of a quadrilateral are equal it is a parallelogram.

77. If the opposite angles of a quadrilateral are equal it is a parallelogram.

78. The diagonals of a parallelogram bisect each other.

79. If the diagonals of a quadrilateral bisect each other it is a parallelogram.

80. If the straight line joining two opposite angles of a parallelogram bisect the angles the four sides of the parallelogram are equal.

81. Draw a straight line through a given point such that the part of it intercepted between two given parallel straight lines may be of given length.

82. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

83. Straight lines bisecting two opposite angles of a parallelogram are either parallel or coincident.

84. If the diagonals of a parallelogram are equal all its angles are equal.

85. Find a point such that the perpendiculars let fall from it on two given straight lines shall be respectively equal to two given straight lines. How many such points are there?

86. It is required to draw a straight line which shall be equal to one straight line and parallel to another, and be terminated by two given straight lines.

87. On the sides AB, BC, and CD of a parallelogram $ABCD$ three equilateral triangles are described, that on BC towards the same parts as the parallelogram, and those on AB, CD towards the opposite parts: shew that the

distances of the vertices of the triangles on AB, CD from that on BC are respectively equal to the two diagonals of the parallelogram.

88. If the angle between two adjacent sides of a parallelogram be increased, while their lengths do not alter, the diagonal through their point of intersection will diminish.

89. A, B, C are three points in a straight line, such that AB is equal to BC: shew that the sum of the perpendiculars from A and C on any straight line which does not pass between A and C is double the perpendicular from B on the same straight line.

90. If straight lines be drawn from the angles of any parallelogram perpendicular to any straight line which is outside the parallelogram, the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles.

91. If a six-sided plane rectilineal figure have its opposite sides equal and parallel, the three straight lines joining the opposite angles will meet at a point.

92. AB, AC are two given straight lines; through a given point E between them it is required to draw a straight line GEH such that the intercepted portion GH shall be bisected at the point E.

93. Inscribe a rhombus within a given parallelogram, so that one of the angular points of the rhombus may be at a given point in a side of the parallelogram.

94. $ABCD$ is a parallelogram, and E, F, the middle points of AD and BC respectively; shew that BE and DF will trisect the diagonal AC.

I. 35 to 45.

95. $ABCD$ is a quadrilateral having BC parallel to AD; shew that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of DC a straight line parallel to AB.

96. $ABCD$ is a quadrilateral having BC parallel to AD, E is the middle point of DC; shew that the triangle AEB is half the quadrilateral.

97. Shew that any straight line passing through the middle point of the diameter of a parallelogram and terminated by two opposite sides, bisects the parallelogram.

98. Bisect a parallelogram by a straight line drawn through a given point within it. depends on 97.

99. Construct a rhombus equal to a given parallelogram.

100. If two triangles have two sides of the one equal to two sides of the other, each to each, and the sum of the two angles contained by these sides equal to two right angles, the triangles are equal in area.

101. A straight line is drawn bisecting a parallelogram $ABCD$ and meeting AD at E and BC at F: shew that the triangles EBF and CED are equal.

102. Shew that the four triangles into which a parallelogram is divided by its diagonals are equal in area.

103. Two straight lines AB and CD intersect at E, and the triangle AEC is equal to the triangle BED: shew that BC is parallel to AD.

104. $ABCD$ is a parallelogram; from any point P in the diagonal BD the straight lines PA, PC are drawn. Shew that the triangles PAB and PCB are equal.

105. If a triangle is described having two of its sides equal to the diagonals of any quadrilateral, and the included angle equal to either of the angles between these diagonals, then the area of the triangle is equal to the area of the quadrilateral.

106. The straight line which joins the middle points of two sides of any triangle is parallel to the base.

107. Straight lines joining the middle points of adjacent sides of a quadrilateral form a parallelogram.

108. D, E are the middle points of the sides AB, AC of a triangle, and CD, BE intersect at F: shew that the triangle BFC is equal to the quadrilateral $ADFE$.

109. The straight line which bisects two sides of any triangle is half the base.

110. In the base AC of a triangle take any point D; bisect AD, DC, AB, BC at the points E, F, G, H respectively: shew that EG is equal and parallel to FH.

111. Given the middle points of the sides of a triangle, construct the triangle.

112. If the middle points of any two sides of a triangle be joined, the triangle so cut off is one quarter of the whole.

113. The sides AB, AC of a given triangle ABC are bisected at the points E, F; a perpendicular is drawn from A to the opposite side, meeting it at D. Shew that the

angle FDE is equal to the angle BAC. Shew also that FDE is half the triangle ABC.

114. Two triangles of equal area stand on the same base and on opposite sides: shew that the straight line joining their vertices is bisected by the base or the base produced.

115 Three parallelograms which are equal in all respects are placed with their equal bases in the same straight line and contiguous; the extremities of the base of the first are joined with the extremities of the side opposite to the base of the third, towards the same parts: shew that the portion of the new parallelogram cut off by the second is one half the area of any one of them.

116. $ABCD$ is a parallelogram; from D draw any straight line DFG meeting BC at F and AB produced at G; draw AF and CG: shew that the triangles ABF, CFG are equal.

117. ABC is a given triangle: construct a triangle of equal area, having for its base a given straight line AD, coinciding in position with AB.

118. ABC is a given triangle: construct a triangle of equal area, having its vertex at a given point in BC and its base in the same straight line as AB.

119. $ABCD$ is a given quadrilateral: construct another quadrilateral of equal area having AB for one side, and for another a straight line drawn through a given point in CD parallel to AB.

120. $ABCD$ is a quadrilateral: construct a triangle whose base shall be in the same straight line as AB, vertex at a given point P in CD, and area equal to that of the given quadrilateral.

121. ABC is a given triangle: construct a triangle of equal area, having its base in the same straight line as AB, and its vertex in a given straight line parallel to AB.

122. Bisect a given triangle by a straight line drawn through a given point in a side.

123. Bisect a given quadrilateral by a straight line drawn through a given angular point.

124. If through the point O within a parallelogram $ABCD$ two straight lines are drawn parallel to the sides, and the parallelograms OB and OD are equal, the point O is in the diagonal AC.

I. 46 to 48.

125. On the sides AC, BC of a triangle ABC, squares $ACDE$, $BCFH$ are described: shew that the straight lines AF and BD are equal.

126. The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

127. The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

128. If the square on one side of a triangle be less than the squares on the other two sides, the angle contained by these sides is an acute angle; if greater, an obtuse angle.

129. A straight line is drawn parallel to the hypotenuse of a right-angled triangle, and each of the acute angles is joined with the points where this straight line intersects the sides respectively opposite to them: shew that the squares on the joining straight lines are together equal to the square on the hypotenuse and the square on the straight line drawn parallel to it.

130. If any point P be joined to A, B, C, D, the angular points of a rectangle, the squares on PA and PC are together equal to the squares on PB and PD.

131. In a right-angled triangle if the square on one of the sides containing the right angle be three times the square on the other, and from the right angle two straight lines be drawn, one to bisect the opposite side, and the other perpendicular to that side, these straight lines divide the right angle into three equal parts.

132. If ABC be a triangle whose angle A is a right angle, and BE, CF be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on BE and CF is equal to five times the square on BC.

133. On the hypotenuse BC, and the sides CA, AB of a right-angled triangle ABC, squares $BDEC$, AF, and AG are described: shew that the squares on DG and EF are together equal to five times the square on BC.

II. 1 to 11.

134. A straight line is divided into two parts; shew that if twice the rectangle of the parts is equal to the sum of the squares described on the parts, the straight line is bisected.

135. Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible.

136. Construct a rectangle equal to the difference of two given squares.

137. Divide a given straight line into two parts such that the sum of the squares on the two parts may be the least possible.

138. Shew that the square on the sum of two straight lines together with the square on their difference is double the squares on the two straight lines.

139. Divide a given straight line into two parts such that the sum of their squares shall be equal to a given square.

140. Divide a given straight line into two parts such that the square on one of them may be double the square on the other.

141. In the figure of II. 11 if CH be produced to meet BF at L, shew that CL is at right angles to BF.

142. In the figure of II. 11 if BE and CH meet at O, shew that AO is at right angles to CH.

143. Shew that in a straight line divided as in II. 11 the rectangle contained by the sum and difference of the parts is equal to the rectangle contained by the parts.

II. 12 to 14.

144. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side and by the straight line intercepted between the perpendicular let fall on it from the opposite angle and the extremity of the base.

145. In any triangle the sum of the squares on the sides is equal to twice the square on half the base together with twice the square on the straight line drawn from the vertex to the middle point of the base.

146. ABC is a triangle having the sides AB and AC equal; if AB is produced beyond the base to D so that BD is equal to AB, shew that the square on CD is equal to the square on AB, together with twice the square on BC.

147. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

148. The base of a triangle is given and is bisected by the centre of a given circle: if the vertex be at any point of the circumference, shew that the sum of the squares on the two sides of the triangle is invariable.

149. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

150. If a circle be described round the point of intersection of the diameters of a parallelogram as a centre, shew that the sum of the squares on the straight lines drawn from any point in its circumference to the four angular points of the parallelogram is constant.

151. The squares on the sides of a quadrilateral are together greater than the squares on its diagonals by four times the square on the straight line joining the middle points of its diagonals.

152. In AB the diameter of a circle take two points C and D equally distant from the centre, and from any point E in the circumference draw EC, ED: shew that the squares on EC and ED are together equal to the squares on AC and AD.

153. In BC the base of a triangle take D such that the squares on AB and BD are together equal to the squares on AC and CD, then the middle point of AD will be equally distant from B and C.

154. The square on any straight line drawn from the vertex of an isosceles triangle to the base is less than the square on a side of the triangle by the rectangle contained by the segments of the base.

155. A square $BDEC$ is described on the hypotenuse BC of a right-angled triangle ABC: shew that the squares on DA and AC are together equal to the squares on EA and AB.

156. ABC is a triangle in which C is a right angle, and DE is drawn from a point D in AC perpendicular to

AB: shew that the rectangle AB, AE is equal to the rectangle AC, AD.

157. If a straight line be drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained by the whole straight line thus produced and the part of it produced is equal to the square on the side of the triangle, shew that the square on the straight line so drawn will be double the square on a side of the triangle.

158. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base: shew that the square on this perpendicular is equal to the rectangle contained by the segments of the base.

159. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base: shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

160. In a triangle ABC the angles B and C are acute: if E and F be the points where perpendiculars from the opposite angles meet the sides AC, AB, shew that the square on BC is equal to the rectangle AB, BF, together with the rectangle AC, CE.

161. Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

III. 1 to 15

162. Describe a circle with a given centre cutting a given circle at the extremities of a diameter.

163. Shew that the straight lines drawn at right angles to the sides of a quadrilateral inscribed in a circle from their middle points intersect at a fixed point.

164. If two circles cut each other, any two parallel straight lines drawn through the points of section to cut the circles are equal.

165. Two circles whose centres are A and B intersect at C; through C two chords DCE and FCG are drawn equally inclined to AB and terminated by the circles: shew that DE and FG are equal.

166. Through either of the points of intersection of two given circles draw the greatest possible straight line terminated both ways by the two circumferences.

167. If from any point in the diameter of a circle straight lines are drawn to the extremities of a parallel chord, the squares on these straight lines are together equal to the squares on the segments into which the diameter is divided.

168. A and B are two fixed points without a circle PQR; it is required to find a point P in the circumference, so that the sum of the squares described on AP and BP may be the least possible.

169. If in any two given circles which touch one another, there be drawn two parallel diameters, an extremity of each diameter, and the point of contact, shall lie in the same straight line.

170. A circle is described on the radius of another circle as diameter, and two chords of the larger circle are drawn, one through the centre of the less at right angles to the common diameter, and the other at right angles to the first through the point where it cuts the less circle. Shew that these two chords have the segments of the one equal to the segments of the other, each to each.

171. Through a given point within a circle draw the shortest chord.

172. O is the centre of a circle, P is any point in its circumference, PN a perpendicular on a fixed diameter: shew that the straight line which bisects the angle OPN always passes through one or the other of two fixed points.

173. Three circles touch one another externally at the points A, B, C; from A, the straight lines AB, AC are produced to cut the circle BC at D and E: shew that DE is a diameter of BC, and is parallel to the straight line joining the centres of the other circles.

174. Circles are described on the sides of a quadrilateral as diameters: shew that the common chord of any adjacent two is parallel to the common chord of the other two.

175. Describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through given point in the given straight line.

III. 16 to 19.

176. Shew that two tangents can be drawn to a circle from a given external point, and that they are of equal length.

177. Draw parallel to a given straight line a straight line to touch a given circle.

178. Draw perpendicular to a given straight line a straight line to touch a given circle.

179. In the diameter of a circle produced, determine a point so that the tangent drawn from it to the circumference shall be of given length.

180. Two circles have the same centre: shew that all chords of the outer circle which touch the inner circle are equal.

181. Through a given point draw a straight line so that the part intercepted by the circumference of a given circle shall be equal to a given straight line not greater than the diameter.

182. Two tangents are drawn to a circle at the opposite extremities of a diameter, and cut off from a third tangent a portion AB: if C be the centre of the circle shew that ACB is a right angle.

183. Describe a circle that shall have a given radius and touch a given circle and a given straight line.

184. A circle is drawn to touch a given circle and a given straight line. Shew that the points of contact are always in the same straight line with a fixed point in the circumference of the given circle.

185. Draw a straight line to touch each of two given circles.

186. Draw a straight line to touch one given circle so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of the latter circle.

187. Draw a straight line cutting two given circles so that the chords intercepted within the circles shall have given lengths.

188. A quadrilateral is described so that its sides touch a circle: shew that two of its sides are together equal to the other two sides.

189. Shew that no parallelogram can be described about a circle except a rhombus.

190. ABD, ACE are two straight lines touching a circle at B and C, and if DE be joined DE is equal to BD and CE together: shew that DE touches the circle.

191. If a quadrilateral be described about a circle the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

192. Two radii of a circle at right angles to each other when produced are cut by a straight line which touches the circle: shew that the tangents drawn from the points of section are parallel to each other.

193. A straight line is drawn touching two circles: shew that the chords are parallel which join the points of contact and the points where the straight line through the centres meets the circumferences.

194. If two circles can be described so that each touches the other and three of the sides of a quadrilateral figure, then the difference between the sums of the opposite sides is double the common tangent drawn across the quadrilateral.

195. AB is the diameter and C the centre of a semicircle: shew that O the centre of any circle inscribed in the semicircle is equidistant from C and from the tangent to the semicircle parallel to AB.

196. If from any point without a circle straight lines be drawn touching it, the angle contained by the tangents is double the angle contained by the straight line joining the points of contact and the diameter drawn through one of them.

197. A quadrilateral is bounded by the diameter of a circle, the tangents at its extremities, and a third tangent: shew that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.

198. If a quadrilateral, having two of its sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to either of the two parallel sides, and terminated by the other two sides, shall be equal to a fourth part of the perimeter of the figure.

199. A series of circles touch a fixed straight line at a fixed point: shew that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

200. Of all straight lines which can be drawn from two given points to meet in the convex circumference of a

given circle, the sum of the two is least which make equal angles with the tangent at the point of concourse.

201. C is the centre of a given circle, CA a radius, B a point on a radius at right angles to CA; join AB and produce it to meet the circle again at D, and let the tangent at D meet CB produced at E: shew that BDE is an isosceles triangle.

202. Let the diameter BA of a circle be produced to P, so that AP equals the radius; through A draw the tangent AED, and from P draw PEC touching the circle at C and meeting the former tangent at E; join BC and produce it to meet AED at D: then will the triangle DEC be equilateral.

III. 20 to 22.

203. Two tangents AB, AC are drawn to a circle; D is any point on the circumference outside of the triangle ABC: shew that the sum of the angles ABD and ACD is constant.

204. P, Q are any points in the circumferences of two segments described on the same straight line AB, and on the same side of it; the angles PAQ, PBQ are bisected by the straight lines AR, BR meeting at R: shew that the angle ARB is constant.

205. Two segments of a circle are on the same base AB, and P is any point in the circumference of one of the segments; the straight lines APD, BPC are drawn meeting the circumference of the other segment at D and C; AC and BD are drawn intersecting at Q. Shew that the angle AQB is constant.

206. APB is a fixed chord passing through P a point of intersection of two circles AQP, PBR; and QPR is any other chord of the circles passing through P: shew that AQ and RB when produced meet at a constant angle.

207. AOB is a triangle; C and D are points in BO and AO respectively, such that the angle ODC is equal to the angle OBA: shew that a circle may be described round the quadrilateral $ABCD$.

208. $ABCD$ is a quadrilateral inscribed in a circle, and the sides AB, CD when produced meet at O: shew that the triangles AOC, BOD are equiangular.

209. Shew that no parallelogram except a rectangle can be inscribed in a circle.

210. A triangle is inscribed in a circle: shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

211. A quadrilateral is inscribed in a circle: shew that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

212. Divide a circle into two parts so that the angle contained in one segment shall be equal to twice the angle contained in the other.

213. Divide a circle into two parts so that the angle contained in one segment shall be equal to five times the angle contained in the other.

214. If the angle contained by any side of a quadrilateral and the adjacent side produced, be equal to the opposite angle of the quadrilateral, shew that any side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

215. If any two consecutive sides of a hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

216. A, B, C, D are four points taken in order on the circumference of a circle; the straight lines AB, CD produced intersect at P, and AD, BC at Q: shew that the straight lines which respectively bisect the angles APC, AQC are perpendicular to each other.

217. If a quadrilateral be inscribed in a circle, and a straight line be drawn making equal angles with one pair of opposite sides, it will make equal angles with the other pair.

218. A quadrilateral can have one circle inscribed in it and another circumscribed about it: shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to each other.

III. 23 to 30.

219. The straight lines joining the extremities of the chords of two equal arcs of a circle, towards the same parts are parallel to each other.

220. The straight lines in a circle which join the extremities of two parallel chords are equal to each other.

221. AB is a common chord of two circles; through C any point of one circumference straight lines CAD, CBE are drawn terminated by the other circumference: shew that the arc DE is invariable.

222. Through a point C in the circumference of a circle two straight lines ACB, DCE are drawn cutting the circle at B and E: shew that the straight line which bisects the angles ACE, DCB meets the circle at a point equidistant from B and E.

223. The straight lines bisecting any angle of a quadrilateral inscribed in a circle and the opposite exterior angle, meet in the circumference of the circle.

224. AB is a diameter of a circle, and D is a given point on the circumference, such that the arc DB is less than half the arc DA: draw a chord DE on one side of AB so that the arc EA may be three times the arc BD.

225. From A and B two of the angular points of a triangle ABC, straight lines are drawn so as to meet the opposite sides at P and Q in given equal angles: shew that the straight line joining P and Q will be of the same length in all triangles on the same base AB, and having vertical angles equal to C.

226. If two equal circles cut each other, and if through one of the points of intersection a straight line be drawn terminated by the circles, the straight lines joining its extremities with the other point of intersection are equal.

227. OA, OB, OC are three chords of a circle; the angle AOB is equal to the angle BOC, and OA is nearer to the centre than OB. From B a perpendicular is drawn on OA, meeting it at P, and a perpendicular on OC produced, meeting it at Q: shew that AP is equal to CQ.

228. AB is a given finite straight line; through A two indefinite straight lines are drawn equally inclined to AB; any circle passing through A and B meets these straight lines at L and M. Shew that if AB be between AL and AM the sum of AL and AM is constant; if AB be not between AL and AM the difference of AL and AM is constant.

229. AOB and COD are diameters of a circle at right angles to each other; E is a point in the arc AC, and EFG is a chord meeting COD at F, and drawn in such a

direction that EF is equal to the radius. Shew that the arc BG is equal to three times the arc AE.

230. The straight lines which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, all intersect at the same point.

231. If two circles touch each other internally, any chord of the greater circle which touches the less shall be divided at the point of its contact into segments which subtend equal angles at the point of contact of the two circles.

III. 31.

232. Right-angled triangles are described on the same hypotenuse: shew that the angular points opposite the hypotenuse all lie on a circle described on the hypotenuse as diameter.

233. The circles described on the equal sides of an isosceles triangle as diameters, will intersect at the middle point of the base.

234. The greatest rectangle which can be inscribed in a circle is a square.

235. The hypotenuse AB of a right-angled triangle ABC is bisected at D, and EDF is drawn at right angles to AB, and DE and DF are cut off each equal to DA; CE and CF are joined: shew that the last two straight lines will bisect the angle C and its supplement respectively.

236. On the side AB of any triangle ABC as diameter a circle is described; EF is a diameter parallel to BC: shew that the straight lines EB and FB bisect the interior and exterior angles at B.

237. If AD, CE be drawn perpendicular to the sides BC, AB of a triangle ABC, and DE be joined, shew that the angles ADE and ACE are equal to each other.

238. If two circles ABC, ABD intersect at A and B, and AC, AD be two diameters, shew that the straight line CD will pass through B.

239. If O be the centre of a circle and OA a radius and a circle be described on OA as diameter, the circum-

ference of this circle will bisect any chord drawn through it from A to meet the exterior circle.

240. Describe a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

241. Describe a circle with a given radius touching a given straight line, such that the tangents drawn to it from two given points in the straight line may be parallel.

242. If from the angles at the base of any triangle perpendiculars are drawn to the opposite sides, produced if necessary, the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the centre of the base.

243. AD is a diameter of a circle; B and C are points on the circumference on the same side of AD; a perpendicular from D on BC produced through C, meets it at E: shew that the square on AD is greater than the sum of the squares on AB, BC, CD, by twice the rectangle BC, CE.

244. AB is the diameter of a semicircle, P is a point on the circumference, PM is perpendicular to AB; on AM, BM as diameters two semicircles are described, and AP, BP meet these latter circumferences at Q, R: shew that QR will be a common tangent to them.

245. AB, AC are two straight lines, B and C are given points in the same; BD is drawn perpendicular to AC, and DE perpendicular to AB; in like manner CF is drawn perpendicular to AB, and FG to AC. Shew that EG is parallel to BC.

246. Two circles intersect at the points A and B, from which are drawn chords to a point C in one of the circumferences, and these chords, produced if necessary, cut the other circumference at D and E: shew that the straight line DE cuts at right angles that diameter of the circle ABC which passes through C.

247. If squares be described on the sides and hypotenuse of a right-angled triangle, the straight line joining the intersection of the diagonals of the latter square with the right angle is perpendicular to the straight line joining the intersections of the diagonals of the two former.

248. C is the centre of a given circle, CA a straight line less than the radius; find the point of the circumference at which CA subtends the greatest angle.

249. AB is the diameter of a semicircle, D and E are any two points in its circumference. Shew that if the chords joining A and B with D and E each way intersect at F and G, then FG produced is at right angles to AB.

250. Two equal circles touch one another externally, and through the point of contact chords are drawn, one to each circle, at right angles to each other: shew that the straight line joining the other extremities of these chords is equal and parallel to the straight line joining the centres of the circles.

251. A circle is described on the shorter diagonal of a rhombus as a diameter, and cuts the sides; and the points of intersection are joined crosswise with the extremities of that diagonal: shew that the parallelogram thus formed is a rhombus with angles equal to those of the first.

252. If two chords of a circle meet at a right angle within or without a circle, the squares on their segments are together equal to the squares on the diameter.

III. 32 to 34.

253. B is a point in the circumference of a circle, whose centre is C; PA, a tangent at any point P, meets CB produced at A, and PD is drawn perpendicular to CB: shew that the straight line PB bisects the angle APD.

254. If two circles touch each other, any straight line drawn through the point of contact will cut off similar segments.

255. AB is any chord, and AD is a tangent to a circle at A. DPQ is any straight line parallel to AB, meeting the circumference at P and Q. Shew that the triangle PAD is equiangular to the triangle QAB.

256. Two circles $ABDH$, ABG, intersect each other at the points A, B; from B a straight line BD is drawn in the one to touch the other; and from A any chord whatever is drawn cutting the circles at G and H: shew that BG is parallel to DH.

257. Two circles intersect at A and B. At A the tangents AC, AD are drawn to each circle and terminated

by the circumference of the other. If CB, BD be joined, shew that AB or AB produced, if necessary, bisects the angle CBD.

258. Two circles intersect at A and B, and through P any point in the circumference of one of them the chords PA and PB are drawn to cut the other circle at C and D: shew that CD is parallel to the tangent at P.

259. If from any point in the circumference of a circle a chord and tangent be drawn, the perpendiculars dropped on them from the middle point of the subtended arc are equal to one another.

260. AB is any chord of a circle, P any point on the circumference of the circle; PM is a perpendicular on AB and is produced to meet the circle at Q; and AN is drawn perpendicular to the tangent at P: shew that the triangle NAM is equiangular to the triangle PAQ.

261. Two diameters AOB, COD of a circle are at right angles to each other; P is a point in the circumference; the tangent at P meets COD produced at Q, and AP, BP meet the same line at R, S respectively: shew that RQ is equal to SQ

262. Construct a triangle, having given the base, the vertical angle, and the point in the base on which the perpendicular falls.

263. Construct a triangle, having given the base, the vertical angle, and the altitude.

264. Construct a triangle, having given the base, the vertical angle, and the length of the straight line drawn from the vertex to the middle point of the base.

265. Having given the base and the vertical angle of a triangle, shew that the triangle will be greatest when it is isosceles.

266. From a given point A without a circle whose centre is O draw a straight line cutting the circle at the points B and C, so that the area BOC may be the greatest possible.

267. Two straight lines containing a constant angle always pass through two fixed points, their position being otherwise unrestricted: shew that the straight line bisecting the angle always passes through one or other of two fixed points.

268. Given one angle of a triangle, the side opposite

it, and the sum of the other two sides, construct the triangle.

III. 35 to 37.

269. If two circles cut one another, the tangents drawn to the two circles from any point in the common chord produced are equal.

270. Two circles intersect at A and B: shew that AB produced bisects their common tangent.

271. If AD, CE are drawn perpendicular to the sides BC, AB of a triangle ABC, shew that the rectangle contained by BC and BD is equal to the rectangle contained by BA and BE.

272. If through any point in the common chord of two circles which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.

273. From a given point as centre describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line may be equal to a given square.

274. Two circles $ABCD$, $EBCF$, having the common tangents AE and DF, cut one another at B and C, and the chord BC is produced to cut the tangents at G and H: shew that the square on GH exceeds the square on AE or DF by the square on BC.

275. A series of circles intersect each other, and are such that the tangents to them from a fixed point are equal: shew that the straight lines joining the two points of intersection of each pair will pass through this point.

276. ABC is a right-angled triangle; from any point D in the hypotenuse BC a straight line is drawn at right angles to BC, meeting CA at E and BA produced at F: shew that the square on DE is equal to the difference of the rectangles BD, DC and AE, EC; and that the square on DF is equal to the sum of the rectangles BD, DC and AF, FB.

277. It is required to find a point in the straight line which touches a circle at the end of a given diameter, such that when a straight line is drawn from this point to the other extremity of the diameter, the rectangle contained

by the part of it without the circle and the part within the circle may be equal to a given square not greater than that on the diameter.

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